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STUDENT'S MISCONCEPTIONS OF SEQUENCES AND SERIES IN SECOND SEMESTER CALCULUS

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DEDICATION

This dissertation is dedicated to my son, Michael, and my wife, Teresa.

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ABSTRACT

STUDENTS' MISCONCEPTIONS OF SEQUENCES AND SERIES IN SECOND SEMESTER CALCULUS

by

David J. Earls

University of New Hampshire, May, 2017

Little is known about the difficulty students have with sequences and series in second semester calculus courses. In my dissertation, I investigate the misconceptions students had about sequences and series as they worked through problems typically seen in a second semester calculus course.

The dissertation begins with a rationale for studying student understanding of sequences and series. Next, I state my goals for the study as research questions. A conceptual framework is then developed for discussing student misconceptions in terms of an optimal concept map. Then, a three-phase methodology is given. In the first phase of the study, I collected data on student solutions to exam problems involving determining the convergence of sequences and series. In phase two, students were interviewed and asked how they would go about solving similar problems. Finally, in phase three, students were given multiple-choice questions on sequences and series.

The results of this study show that students lack the prerequisite skills needed to be successful in second semester calculus. In addition, students have difficulty selecting which series test to use when investigating series convergence, why the assumptions of such tests are important, and what the conclusions in series tests tell them.

The dissertation concludes with a discussion of errors versus misconceptions, how this study contributes to the existing literature on sequences and series, limitations of the study, and implications for further research.

Introduction

Researchers have argued that students have difficulty in first semester calculus courses because they lack the necessary prerequisite knowledge (Ferrini-Mundy & Graham, 1991; Carlson, Madison & West, 2010; Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997). In particular, many students entering a first semester calculus course struggle with the course because they have a weak understanding of the function concept (Ferrini-Mundy & Graham, 1991; Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997).

When students finish their first semester calculus course and enter second semester calculus, the expectation is that they have acquired the necessary prerequisite knowledge, such as an understanding of the function concept, from their prior courses. For example, before teaching students sequences and series, concepts that are critical parts of the calculus sequence that experts think are essential for students to understand (Sofronas, DeFranco, Vinsonhaler, Gorgievski, Schroeder, & Hamelin, 2011), instructors might assume students have an understanding of the function concept. As another example, linear algebra students are expected to have a knowledge of set theory, and research has shown that a lack of knowledge about set theory causes difficulty for students in linear algebra (Dogan-Dunlap, 2006). Based on exercises and theorems covered in previous chapters leading up to a chapter on sequences and series in calculus textbooks that the researcher has used, prerequisite knowledge needed for second semester calculus includes the concept of function, trigonometric functions, derivatives, integrals, limits, infinity, and aspects of algebraic manipulation (Hass, Weir, & Thomas, 2012; Hughes-Hallet, Gleason, McCallum, et al., 2005; Finney & Thomas, 1994).

However, many of my students in second semester calculus seemed to lack the prerequisite knowledge needed to succeed in many areas and in particular in the area of sequences and series. For example, when using the ratio test, some of the researcher's students were unable to correctly simplify a rational expression. Students' struggles in this particular area of the course provided motivation to investigate the existing literature on student understanding of sequences and series, and the role prerequisite knowledge plays in that understanding. Looking at the literature, there appears to be little research on student understanding of sequences and series. The literature on student understanding of sequences and series focuses on the role infinity plays on students' understanding of series (Sierpińska, 1987), student understanding of definitions (Roh, 2008; Martínez-Planell, R., Gonzalez, A., DiCristina, G., & Acevedo, V.), learner's beliefs about their role as a learner and the relationship between these beliefs and approaches to solving convergence problems (Alcock & Simpson, 2004; Alcock & Simpson, 2005), how series are introduced to students (González-Martín, Nardi, & Biza, 2011), and the difficulties students have accepting that comparison tests can be inconclusive (Nardi and Iannone, 2001). However, there is a gap in the existing literature on sequences and series regarding the role prerequisite knowledge plays in student understanding of these concepts. Moreover, the participants in many of these studies are not second semester calculus students, but undergraduates in real analysis (González-Martín, Nardi, & Biza, 2011; Alcock & Simpson, 2004; Alcock & Simpson, 2005), graduate students (Martínez-Planell, Gonzalez, DiCristina, & Acevedo, 2012), and humanities students (Sierpińska, 1987). Consequently, research is needed to investigate the difficulties second semester calculus students have with sequences and series.

This dissertation study tries to understand what difficulties arise for students studying sequences and series in second semester calculus, and in what ways those difficulties relate to prerequisite knowledge they are assumed to have entering the course, thus potentially contributing to the literature in this area. In addition, the study will investigate the difficulties students have that are connected to more than just a lack of prerequisite knowledge. In the next section, these goals are stated as research questions.

Research Questions

From work with students, a pilot study, and the existing literature, the following research questions are proposed:

1. What misconceptions of sequences and series are revealed when students solve problems on sequences and series typically seen in a second semester calculus course?
2. In what ways do these misconceptions relate to the prerequisite knowledge students are expected to have prior to starting a second semester calculus course?
3. What additional understanding or conceptualization of sequences and series might students need to be successful in second semester calculus courses?

The first research question focuses primarily on the errors that students make as they solve problems. The misconceptions that are evident as students attempt to solve problems will be compared against an optimal concept map for sequences and series developed from the researcher's teaching experiences as well as two calculus textbooks. This comparison will be useful to see ways these misconceptions show potential gaps in the relational understanding of sequences and series. Concept maps as well as the notions of concept image and relational understanding will be discussed in the conceptual framework in chapter one.

The second research question was developed because, as mentioned above, a lack of prerequisite knowledge is one reason for student misconceptions in first semester calculus (Ferrini-Mundy & Graham, 1991; Carlson, Madison & West, 2010; Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997). The second question is asking if a lack of prerequisite knowledge helps explain why students have difficulty with sequences and series in second semester calculus.

The third research question focuses on what additional knowledge, if any, students need to solve problems. For example, a student might have difficulty choosing the correct test for convergence or divergence, fail to check the assumptions of a test, or have difficulty interpreting the conclusion of a convergence test.

The second and third questions also investigate the reasons students give for why they are making mistakes. For example, a student that forgets to include a series sign when indicating that $\sum \frac{1}{n}$ diverges might have just made a notational error. It is also possible that the student does not understand the difference between the sequence $\frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$. Only by interviewing students and asking them questions as they work through problems can it be determined why a particular error was made and determine if this error was a result of a lack of prerequisite knowledge.

A pilot study was conducted and provided some insight into each of these questions. Interviewing students as they worked through typical second semester calculus problems began to reveal how a lack of prerequisite knowledge can lead to difficulty solving problems on sequences and series. In addition, the pilot study begins to reveal how a lack of knowledge about sequences and series aside from prerequisite knowledge causes difficulty for students when working through problems. The results from the full study provide more detailed answers than the discoveries in the pilot study.

In the next few chapters, the study is outlined in more detail. Chapter one begins with a discussion of the theoretical perspective. Then, the conceptual framework for the study is developed, and this framework will include the notions of concept image and concept maps. The chapter concludes with a review of relevant literature on sequences and series as well as what is already known about student understanding of concepts and skills needed to be successful in second semester calculus, such as the function concept, and how these skills relate to sequences and series.

Chapter two describes the methodology for the study. It begins with a description of participants and the setting of the study. It will indicate how data was be collected, as well as how the data helped answer each of the research questions. The chapter also describes how the data was analyzed, using the pilot study as an example.

Chapter three describes the qualitative results of the study, while chapter four states the results of the quantitative analysis. This dissertation concludes in chapter five with a discussion of the results and implications for further research.

Chapter 1: Theoretical Perspective, Conceptual Framework, and Literature Review

In this chapter, the theoretical perspective for this study is presented, followed by a description of the study's conceptual framework. The chapter concludes with a review of the relevant literature.

Theoretical Perspective

Skemp (1976/2006) distinguishes between two different types of understanding when learning mathematics. The first is instrumental understanding. Instrumental understanding of mathematics is described by Skemp as "rules without reasons" (p. 89). An example of instrumental understanding in mathematics would be knowing that, when dividing by a fraction, one should instead multiply by the reciprocal. A student (or teacher) with an instrumental understanding knows this rule and can use it effectively, but does not understand why this rule works.

The second type of understanding is referred to by Skemp (1976/2006) as relational understanding. Relational understanding of mathematics is described by Skemp as "knowing both what to do and why" (p. 89). Someone with a relational understanding of mathematics would know the rule for dividing by a fraction, but would also know why this rule works.

Skemp (1976/2006) notes that there are benefits to both instrumental and relational understanding. Skemp states that instrumental understanding is easier to develop, has more immediate and apparent rewards, and one can get the right answer faster and more reliably. Relational understanding, on the other hand, is adaptable to new problems, is easier to develop since mathematics is seen as a connected whole instead of many separate rules, and is organic in the sense that, when someone understands a mathematical concept relationally, he will go out and seek new areas to explore.

Skemp (1976/2006) argues for relational understanding over instrumental understanding. "The kind of learning which leads to instrumental mathematics consists of the learning of an increasing number of fixed plans…In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans…" (p. 94-95).

The theory of constructivism is consistent with this notion of building up a conceptual structure, or schema, in relational mathematics described by Skemp (1976/2006). Constructivists believe that knowledge and reality are constructed by individuals (von Glasersfeld, 1996; Hatch, 2002), with the requirement that the knowledge is "viable, that it fits into the world of the knower's experience" (von Glasersfeld, p. 310). Constructivism, "recognizes that knowing is active, that it is individual and personal, and that it is based on previously constructed knowledge" (Ernest, 2006, p. 3).

Conceptual Framework

The purpose of this section is to develop a framework for the study that is consistent with a constructivist ideology. The notions of concept image, concept definition, and concept maps are introduced.

Tall and Vinner (1981) use the term "…*concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). A student's concept image of a particular concept usually results from working with examples and non-examples (Vinner & Dreyfus, 1989). The concept image for concepts that do not have graphical components or have weak graphical components will include symbols, formulas, and associated properties (Vinner & Dreyfus, 1989). As an example, Vinner and Dreyfus (1989) note that the function concept has both graphical and non-graphical components.

Tall and Vinner (1981) define *concept definition* as "…a form of words used to specify that concept" (p. 152). Here, they also differentiate between a student's *personal* concept definition and a *formal* concept definition. A formal concept definition is the definition of a concept that is agreed upon by the mathematical community. A personal concept definition, however, might be constructed by the student and might change over time.

The relationship between concept image and concept definition is described by Tall and Vinner (1981). "For each individual a concept definition generates its *own* concept image" (p. 153). In other words, the words used to describe a particular concept generate a mental image associated with the concept. As an example, they consider the function concept. The formal concept definition of function can be described as a relation between two sets where each element of the first set is assigned exactly one element of the second set. However, a student studying function might not remember this definition, and the concept image for the student might include the idea that a function must be given by a rule or formula.

Because concept images are mental constructions, it can be hard to know exactly what a concept image for a particular concept looks like for a given student. Even concept maps cannot completely reveal a student's concept image (Williams, 1998). A concept map, when constructed by an individual such as a student, is "a direct method of looking at the organization and structure of an individual's knowledge within a particular domain and at the fluency and efficiency with which the knowledge can be used" (p. 414). Williams (1998) explains that concept maps do provide some insight into a student's conceptual understanding, even if they don't reveal concept images.

Summary. The notions of concept image, concept definition, and concept maps serve as a conceptual framework for this study. Because concept images are mental constructions, this terminology can be used to describe existing student knowledge. When classifying student misconceptions of sequences and series, these misconceptions will be referred to in terms of the glimpses they provide into a student's concept image and relational understanding.

Literature review

This section reviews the relevant literature on sequences and series as well as literature dealing with prerequisite knowledge students are expected to have entering second semester calculus courses. It is important to review literature on prerequisite knowledge because a lack of prerequisite knowledge can cause difficulties for students. In particular, linear algebra students are expected to have a knowledge of set theory, and research has shown that a lack of knowledge about set theory causes difficulty for students in linear algebra (Dogan-Dunlap, 2006).

The section begins with a review of the literature involving the function concept, limits, and differentiation and integration. The review concludes with a synthesis of literature on sequences and series.

Function. Clement (2001) found that some student concept images of functions consist of requiring that a function be defined by a single rule. For example, a piecewise function can be considered in a student's concept image as two or more functions (Clement, 2001; Carlson & Oehrtman, n.d.). This is, of course, a misconception, as a piecewise function is one function defined on a “split domain” (Clement, 2001, p. 745). A study done by Markovits, Eylon, and Bruckheimer (1986) found that students had difficulty answering questions that dealt with piecewise functions. In this study, 400 ninth grade students were given a wide variety of problems dealing with graphical and algebraic representations of functions. When presented with an algebraic representation of a piecewise function, about 50% of the students surveyed said that the piecewise function was not a function.

Clement (2001) found that another aspect of student concept images of function is that a function’s graph needs to be continuous. As an example, consider the greatest integer function, the graph of which is a step function and not continuous. Students who think the graph of a function must be continuous would not classify the greatest integer function as a function. Other difficulties with graphs of functions include confusing distance traveled with the distance from an object (Kerslake, 1981; Van Dyke and White, 2004), identifying the graph of a linear function as constant simply because it has a constant rate of change, and difficulty seeing a point on a graph as a solution to its corresponding equation (Van Dyke and White, 2004). Students also have difficulty writing an equation for a line given a graph of a linear function with clearly labeled x- and y-intercepts (Carpenter, Corbit, Kepner, Lindquist, & Reys, 1981).

Researchers have found that students have difficulty identifying constant functions as functions (Clement, 2001; Carlson & Oehrtman, n.d; Vinner & Drefus, 1989), and they hypothesize that there are several reasons for this. First, concept images of students can include the requirement that a function is one-to-one, that is to say, only one element in the domain corresponds to a certain value in the range (Clement, 2001). Since a constant function is not one-to-one, students with this concept image will not view constant functions as functions (Clement, 2001). Second, students may also think that constant functions are not functions because they do not vary (Carlson & Oehrtman, n.d.) Finally, a study done by Vinner and Dreyfus (1989) confirms that students may incorrectly identify constant functions as linear maps. In this study, students were asked, "Does there exist a function all of whose values are equal to each other?" (p. 359). Only 55% of the students surveyed (271 college students and 36 junior high school teachers) answered the question correctly. An incorrect answer that occurred frequently was the function f(x) = x. This is consistent with the misconception students have identifying graphs of linear function as constant functions described by Van Dyke and White (2004).

Some students have difficulty interpreting the symbols used in functional notation (Sajka, 2003). For example, in a case study done of one secondary school student, Sajka (2003) found that this student had difficulty understanding the notation f(3), associating f(3) with the zero of the function, and that this student thought of f(x), f(y), and f(x + y) as three different functions. Sajka (2003) also found that this student could not distinguish between the concept of function and the concept of the formula of a function.

Students also have difficulties understanding trigonometric functions. Weber (2005) found that only five out of 31 students in a traditionally taught trigonometry class in college were correctly able to approximate sin 340 degrees. Only 9 students were able to determine when $\sin\theta$ is decreasing and why. Only 4 students could explain why the trigonometric identity $(\sin\theta)^2 + (\cos\theta)^2 = 1$ is true. After interviewing 4 of these students, Weber concluded that students that had received the traditional instruction were unable to create geometric figures to answer questions about trigonometric functions and only had an instrumental understanding of the sine function.

Carlson and Oehrtman (n.d.) explain that students have difficulty understanding functions in terms of input values and output values. For example, some students that received an 'A' in college algebra tried to evaluate the expression f(x + a) by adding 'a' to the end of the expression for f.

Carlson and Oehrtman (n.d.) note that students have difficulty using function notation to write a functional relation. For example, many precalculus students did not know that, if asked to write s as a function of t, they needed to write something of the form s = expression involving t. Also, in the case of a basic function like f(x) = 3x, students did not know what each of f, f(x), and 3x represent.

Researchers have provided a few hypotheses for why students have so much difficulty with the function concept. Clement (2001) believes that not enough time is spent discussing the definition of function and the relationships between the different ways of representing functions. Clement (2001) and Sajka (2003) feel assessment tools inadequately test student understanding of function, and so students can do well in school without understanding the function concept. Van Dyke and White (2004) argue that curriculums do not place enough emphasis on the graphical representation of functions. Carlson and Oehrtman (n.d.) feel that schools place too much emphasis on instrumental understanding, such as manipulating equations, and not enough emphasis on relational understanding.

In summary, there are many misconceptions of function, and students have great difficulty in dealing with the notion of function. This study seeks to provide evidence that difficulties with the function concept are one reason that students struggle solving problems on sequences and series.

Limits. In the previous section, research that examined the difficulty students have with the function concept was discussed. Because problems regarding sequences and series in second semester calculus typically involve convergence, it is important to review the literature on the difficulties students have with the concept of limit. Though many of these difficulties appear to show a lack of relational understanding, and students only need to perform computations in second semester calculus, it is important to review these difficulties because it may provide some insight into student thinking about evaluating limits.

Research indicates that one difficulty that students have with limits is that they do not understand the difference between limit and the value of a function (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996). In their study on student understanding of the limit concept, Cottrill et al. (1996) gave an instructional treatment of limits centered around APOS theory over the course of several weeks to a first semester calculus course. They found that students, when asked to find the limit of a function at a point *a*, instead gave the value of the function at *a*.

Some students think that a function can never reach its limit (Williams, 1991). In his study, Williams (1991) asked 341 second semester calculus students to determine whether the following statement is true or false: "A limit is a number or point the function gets close to but never reaches" (p. 221). Of the 341 students surveyed, 70% said that this statement was true. Williams (1991) offers an explanation for this, saying that students often choose the easiest view of limit rather than a correct one. In other words, students adapt this view of limit, though incorrect, because it is easier for them to understand.

Students have difficulty with the formal definition of limit (Williams, 1991; Cottrill et al., 1996). Williams (1991) interviewed ten of the 341 students in his original study, and after five instructional sections with them, none of the students adopted the formal definition. None of the students in the limit study done by Cottrill et al. (1996) mentioned in the previous paragraphs were able to apply the formal definition to solve problems, though a few students at least appeared to understand the limit as an object in APOS theory. Cory and Garofalo (2011), in their study of the effects of software on three preservice teachers understanding of the limit concept, found that dynamic software helped improve the understanding of the definition of limit for these three preservice teachers. The guided reinvention process helped two students improve their understanding of the definition of limit of a sequence in a study by Oehrtman, Swinyard, and Martin (2014). In this study, a teaching experiment involving the process of guided reinvention was done with two students that had not seen the formal definition of convergence. At the start of the study, the students struggled to understand the formal definition of a limit. By the end of the guided reinvention process, the students adopted views of limit that were more consistent with the formal definition.

Szydlik (2000) found that another misconception of the limit concept is that a limit is a bound that cannot be crossed. Students that hold this misconception would not be able to recognize limits of functions that oscillate around a value, but get closer to that value. Note that this differs slightly from the notion of a limit as unreachable, for a bound can be reached but not crossed. Still, Szydlik (2000) cautions both notions are serious misconceptions because the ideas of a limit as a bound and a limit as unreachable contradict the formal definition of limit.

One reasons students have such difficulty with the limit concept is that many students rely on external sources of conviction (Szydlik, 2000). A source of conviction for an individual is a feeling that individual has about how mathematical facts are confirmed. Students with an external source of conviction appeal to external sources, such as teachers and textbooks, to validate mathematical facts. In other words, a student with an external source of conviction may argue that a theorem is true simply because it appears in a textbook. This is in contrast with individuals with an internal source of conviction that appeal to internal sources such as evidence, logic, and reasoning. In other words, a student with an internal source of conviction could argue about the existence of a limit based on the definition of the limit, or by using theorems about limits.

A study of 27 university calculus students and their ability to understand the limit concept done by Szydlik (2000) indicates how sources of conviction relate to difficulty with the limit concept. In this study, Szydlik (2000) shows that many students with an external source of conviction could not give a satisfactory definition of the limit of a function. In addition, these students with an external source of conviction could not explain why the methods they used to solve limit problems were valid. Most of these students, though Szydlik (2000) doesn't state specifically how many, believed that limits were bounds or unreachable. In contrast, students with internal sources of conviction, such as appealing to logic and reasoning, gave acceptable definitions of limit and did not see limits as bounds or unreachable.

Another possible reason for the difficulty with the limit concept is that students don't see the need for the formal definition of the limit (Williams, 1991). Williams (1991) argues that, in the eyes of the students, work they typically do in a classroom does not require more than knowledge of graphing and simplistic views of limit. Thus, there is no motivation for students to learn the more formal definition.

In summary, students have many difficulties when first introduced to the limit concept. In this study, weak understanding of the limit concept may be related to students' difficulties determining the convergence of sequences.

Differentiation and Integration. Differentiation and integration play key roles when students are determining the convergence of sequences and series. For example, L'Hôpital's rule can be used in some circumstances to determine if a sequence converges, and using L'Hôpital's rule involves taking derivatives. The integral test can sometimes be used to determine if a series converges. The literature indicates that students have many difficulties understanding differentiation and computing derivatives. In one study, Orton (1983) did task-based interviews with 110 students ages 16-22. These students were given tasks about rates of change and derivatives. Orton found that students had difficulties with basic algebra, such as solving a basic quadratic and expanding a binomial expression, computing average rates of change, understanding the relationship between average rate of change and instantaneous rate of change, with symbols such as dy and dx, differentiating ${}^{2}/{}_{x^2}$, and understanding the meaning of rates of change that were negative or zero. Judson and Nishimori (2005) interviewed 26 BC calculus students in America and 18 calculus students in Japan. They found that, when told that the derivative of a function was identically zero, both American and Japanese high school students responded that the function must have a critical point, rather than stating that the function is constant. Judson and Nishimori (2005) also noted that American students had a difficult time simplifying any expressions that contained a radical sign when trying to simplify expressions after taking derivatives.

Research indicates that students also have numerous difficulties understanding integration and computing integrals. Orton (1983) discovered that students had trouble using the power rule for integration on expressions with negative or fractional exponents, had difficulty integrating constants, made arithmetic errors when computing a basic definite integral, could not explain why an integral needed to be split up if parts of the curve lied above the axis and other parts below the axis, did not break up the improper integral $\int_{-1}^{2} \frac{1}{x^2}\, dx$ at 0, and did not interpret integration as the limit of a sum. Judson and Nishimori (2005), found that both American and Japanese high school students did not know they needed to use the chain rule to find the derivative of $\int_{0}^{1/x} \frac{1}{1+t^2}\, dt$. Moreover, many students did not think of $\int_{0}^{x} \frac{1}{1+t^2}\, dt$ as a function.

In summary, students have numerous difficulties understanding and computing derivatives and integrals. Differentiation is important when solving convergence problems in second semester calculus because L'Hôpital's rule can be used to help determine if a sequence converges, and integration is important for use in the integral test. Consequently, errors in computing derivatives and integrals could lead to errors in determining convergence.

Existing Literature on Misconceptions of Sequences and Series. Przenioslo (2006) discovered that many students do not think of a sequence as a function. This is in direct contrast to textbooks that define a sequence as a function (Hass, Weir, & Thomas, 2012). In Przenioslo's (2006) study of 446 secondary school students and 156 students beginning university level work, only 12% thought of a sequence as a function. Przenioslo (2006) also noted some other misconceptions of the concept of sequence in his study. In particular, he noticed that some students thought a sequence must be monotone (12%), the terms of a sequence must be described by an explicit formula (11%), and the difference between sequential terms in the sequence must always remain the same (7%).

The literature indicates that another difficulty that students encounter with sequences is a difficulty to correctly use definitions, such as the definition of convergence (Alcock & Simpson, 2004; Alcock & Simpson, 2005). In their study, Alcock and Simpson (2004) found that students who were visual learners, or those that relied on visual reasoning such as graphs to understand concepts, and appealed to external sources of authority, such as textbooks, as opposed to internal sources of authority, such as evidence and reasoning, were less likely to understand definitions and could not use definitions to make arguments. The same result held for students that tended toward non-visual reasoning, such as using algebraic symbols (Alcock & Simpson, 2005). Thus, students with an internal source of authority appear to be more likely to understand definitions such as convergence (Alcock & Simpson, 2004; Alcock & Simpson, 2005).

Roh (2008) also explains the difficulties students have determining convergence of sequences. In her study, Roh (2008) found that student concept images of limit from prior calculus courses had an impact on their view of the definition of convergence and their ability to correctly determine whether a sequence converged. In particular, students that had a view of limit as cluster points or asymptotes had difficulty determining whether some sequences converged or diverged.

In addition to struggling to understand what it means for a sequence to converge, research shows that students also struggle with the definition of series convergence (Martínez-Planell, Gonzalez, DiCristina, & Acevedo, 2012). In their study of ten graduate students, Martínez-Planell et al. (2012) noticed that one graduate student correctly answered that the limit of partial sums and the sum of an infinite series were the same thing. However, when probed further, the student changed his answer and disregarded the definition of series convergence as a limit of partial sums. The authors state the reason for this change was that the student had simply memorized the definition, and so answered the question with a memorized fact. However, when presented with another problem, the student tried to determine the sum of the series by first adding all the terms (or as many as he could to determine a pattern), and then by taking the limit of the partial sums. When the student got two different answers, he rejected the notion of the sum of the series being the same as the limit of partial sums. In general, when solving problems regarding infinite series, Martínez-Planell et al. (2012) noticed that students tend to rely on properties regarding finite sums, rather than looking at the limit of partial sums.

Sierpińska (1987) notices one difficulty students have finding sums of infinite series is that they struggle with the concept of infinity. In her study, Sierpińska (1987) presented students with the result and proof that 0.999… = 1. The proof presented was the algebraic proof, where x = 0.999…, so 10x = 9.999…, and thus 9x = 9 so x = 1. Some students thought the result and proof were incorrect, some thought the result was incorrect but the proof was correct, some thought the result was correct but the proof incorrect, some accepted the result and the proof but interpreted the proof incorrectly, and one student accepted both the proof and result and interpreted it correctly. Sierpińska (1987) attributed the difficulty students had with the result and proof to their understanding of the concept of infinity. In particular, some students had a notion of infinity as something that can be reached. These students accepted the result because in their minds, they can reach infinity, though it is unlikely according to Sierpińska (1987) that such students would accept the given proof. Other students felt that infinite processes can never be completed, and thus 0.999… is not a number, and therefore cannot possibly be equal to one.

Nardi and Iannone (2001) discovered that students have difficulty accepting that tests for convergence can be inconclusive. In their study, 60 calculus students were asked if certain series converged. In one problem, the students were asked if $\sum_{n=1}^{\infty} \frac{n+2}{n^3-n^2+11}$ converged or diverged.

One student used the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and correctly obtained a limit of zero. However, instead of saying that the test was inconclusive in this instance, the student incorrectly concluded the original series does not converge. The authors also state that several of this student's peers made similar mistakes. Only 16 students' solutions (out of 60) were collected for this question. Of those 16, only seven answered the question correctly.

In summary, students have difficulty recognizing a sequence as a function (Przenioslo, 2006) and have difficulty understanding definitions when studying sequences, such as the definition of convergence (Alcock & Simpson, 2004; Alcock & Simpson, 2005; Roh, 2008). Students struggle with understanding what it means for a series to converge (Martínez-Planell et al. 2012), have difficulties with the concept of infinity (Sierpińska, 1987), and have trouble accepting that a convergence test may be inconclusive (Nardi & Iannone, 2001). Though in this study students are not presented with formal definitions of sequence convergence, many of the misconceptions of sequences and series found in the literature can be seen in this study as students try to solve problems on sequences and series in second semester calculus courses.

Summary and Conclusion

The focus of this chapter was on establishing a framework for a study on second semester calculus students' misconceptions of sequences and series, and analyzing existing literature on student difficulty with concepts closely related to sequences and series. This study seeks to build on the existing literature by finding out in what ways misconceptions of these concepts effects students' abilities to solve problems on sequences and series in second semester calculus. It also seeks to support the findings of previous studies regarding student misconceptions of sequences and series.

The focus on the next chapter is on designing the methodology for this study. The chapter will begin with a description of grounded theory, and then a three-phase description of the methodology for the study will be presented.

Chapter 2: Methodology

The purpose of this chapter is to describe the methodology for this study. It includes a description of the context and participants, data collection, and data analysis. Included in this description will be examples from a pilot study as well as the construction of an optimal concept map.

This study proceeds in three phases, so a careful description of each phase is provided. The first two phases utilized qualitative methodology. The final phase uses the qualitative data collected in the first two phases to develop a multiple-choice assessment that was examined quantitatively. Because the methodology used in the first two phases was inspired by grounded theory, a brief description of grounded theory and the rationale for this approach is given below.

Grounded Theory

Creswell (2013) describes grounded theory as, "a qualitative research design in which the inquirer generates a general explanation (a theory) of a process, an action, or an interaction shaped by the views of a large number of participants" (p. 83). In grounded theory, the researcher starts by thinking about a particular process he would like to investigate. Ultimately, the researcher is hoping that he will come up with a theory that might help explain this process. Data collection most often involves interviews. Data analysis in grounded theory is highly structured, and it involves three rounds of coding. In the first round of coding, the researcher uses open coding to develop codes that serve as the basis for the theory. As categories are developed in the open coding, they may be merged and renamed. In the second round of coding, called axial coding, connections are made between the categories that developed in the first round of coding. Finally, in the third round of coding, called selective coding, the intersection of all categories is taken to create the theory.

It is important to note that the qualitative components in this study are not true grounded theory. In true grounded theory, there is no theoretical or conceptual framework to start with, and a literature review is not done until after data analysis is complete. Thus, the optimal concept map described in the next section would be entirely generated from the data. But in this study, the optimal concept map is based on the experiences of experts in the field and calculus textbooks. However, it is likely that the optimal concept map is incomplete. There may be other concepts that should be included in the optimal concept map, and those concepts will likely be grounded in the data. A grounded theory approach thus serves to help ensure that the optimal concept map created by experts for this study captures what the ideal student should know.

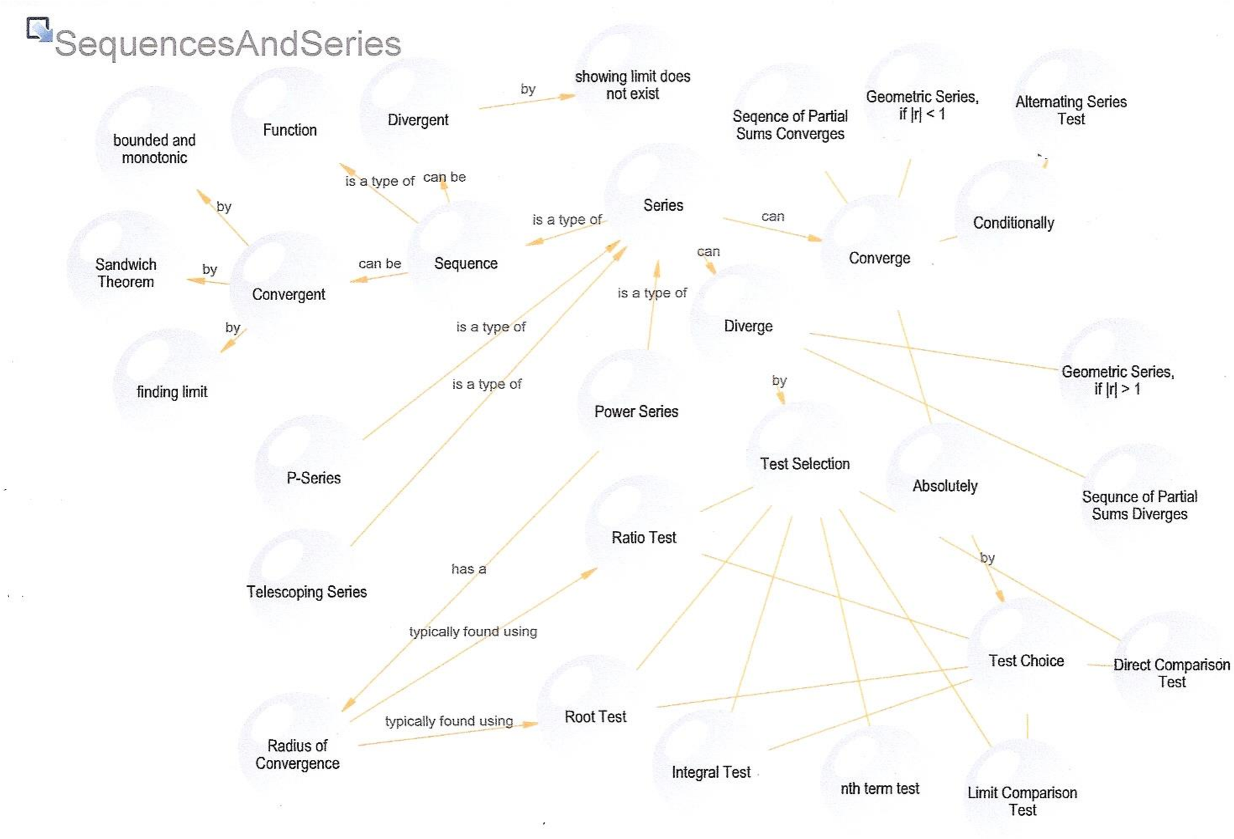
Using a grounded theory approach, in addition to potentially adding to the conceptual framework, addresses the research questions. In particular, by learning about student misconceptions and the relationship between these misconceptions and prerequisite knowledge students are expected to have, questions about what students *should* know can be answered.

Optimal Concept Map

Williams (1998) argues that concept maps can be used to depict mathematical knowledge. Though having students draw concept maps regarding sequences and series would provide insight into their mathematical knowledge of sequences and series, the research questions stated in the introductory chapter are asking about misconceptions revealed when solving problems. In other words, this study focused more on students' problem solving knowledge rather than declarative knowledge. Consequently, rather than have students draw concept maps, an *optimal concept map* was constructed. In the context of this study, an *optimal concept map* is a map that is representative of what the ideal second semester calculus student should know about sequences, series, and prerequisite skills upon completion of the course. Errors that are revealed when students solve problems were then compared against this optimal concept map during selective coding.

The optimal concept map is given in figure 3.0 and can also be found in Appendix A. To make sure it is truly representative of what a second semester calculus student should know about sequences and series at the end of the course, the map was created based on the researcher's experience teaching the course as well as two calculus textbooks. These two textbooks are, *Calculus: Single and Multivariable* (Hughes-Hallet, Gleason, McCallum, et al., 2005), which the researcher used when teaching Calculus I in high school, and *University Calculus: Early Transcendentals* (Hass, Weir, & Thomas, 2012), which has been used by instructors and teaching assistants at the university where this research took place. In Hass, Weir, and Thomas (2012), sequences and series are covered in chapter nine, and many of the exercises are algorithmic. This book was chosen because it is the textbook used at the University where this research took place. Hughes-Hallet et al. (2005) is somewhat different. There seems to be more of an emphasis on relational understanding, that is, problems that focused more on concepts than algorithmic procedures, in this text, and it was chosen for this reason.

Figure 3.0: Optimal Concept Map



Once the optimal concept map was generated, the map was presented to three experts in the field that had experience teaching second semester calculus as an instructor or a teaching assistant. The purpose of giving the map to experts was to further ensure that the map was representative of the ideal second semester calculus student and what that student would know at the end of the course. The map is discussed in more detail in the following sections. The first section discusses the sequences portion of the map, and the second section discusses series.

Sequences. The purpose of this section is to provide details on the development of the nodes branching off from the ‘sequence’ node in the optimal concept map. In particular, it describes the development of the nodes on the lower half of the map that have arrows coming from the ‘sequence’ node.

Both textbooks mentioned in the previous section refer to sequences as a type of function. Concept maps can be generated for the function concept, and an example can be found in Appendix B. Note that students that take second semester calculus should have completed courses in precalculus and first semester calculus. Consequently, much of what is seen in the concept map of function is considered prerequisite knowledge for a second semester calculus course. This also explains the 'function' node in the optimal concept map.

Typically, problems that students need to solve involve determining whether or not a sequence given by an explicit formula converges or diverges. Convergence can be determined in one of several ways. Students can evaluate the limit of the sequence as "n" goes to infinity. Some prerequisite knowledge that might help students here includes the use of L'Hôpital's rule. Students could also use the Sandwich Theorem to determine sequence convergence. The textbooks also indicate that students could show that a sequence is both bounded and monotonic (if the sequence fits this description), though few students try this approach. This is why there is a 'convergent' node as well as the nodes that branch from this node.

Students have fewer options to show that a sequence diverges. In this case, they need to show that a limit does not exist. On exams, students may be asked to classify this divergence as $\pm\infty$ where appropriate. Hence, there is a 'divergent' node.

Series. The purpose of this section is to provide details on the development of the nodes branching off from the 'series' node in the optimal concept map. In particular, it describes the development of the nodes on the upper half of the map that have arrows coming from the 'series' node.

All three textbooks are again consistent in referring to series as sequences of partial sums. In particular, they mention that saying a series converges is equivalent to saying that the sequence of partial sums converges. Similarly, if the sequence of partial sums is divergent, then the series diverges. This is why the 'series' node is connected to the 'sequence' node.

Typically, problems that students need to solve involve determining whether a series converges or diverges. Unlike with sequences, students have many options for determining convergence or divergence. 'Converges' and 'diverges' are each nodes on the optimal concept map.

If a student feels that a series converges conditionally, they can make use of the alternating series test. If a student feels that a series converges absolutely, or if all the terms in the series are positive to begin with, he can use a comparison test (direct or limit), the ratio test, the root test, or the integral test. Alternatively, if the series is a geometric series, $\sum_{n=1}^{\infty} ar^{n-1}$, a student could show that |r| < 1, and hence the series converges. Moreover, the student can show that the series converges to $\frac{a}{1-r}$. Finally, the student can also show that a series converges by computing the limit of the partial sums, as in the case of a telescoping series. Each test has its own node in the optimal concept map.

Should a student feel that a series diverges, they have a number of tests that they can use to show divergence. Students could use the nth term test for divergence, a comparison test (direct or limit), the ratio test, the root test, or the integral test. Alternatively, if the series is a geometric series, $\sum_{n=1}^{\infty} ar^{n-1}$, a student could show that $|r| \geq 1$, and hence the series diverges. Finally, the student can also show that a series diverges by showing that the limit of the partial sums does not exist, though in practice this rarely happens because in many cases it is difficult to define a formula for the partial sums. Each test for divergence also has a node on the optimal concept map.

Another type of problem students encounter involving series is to find the radius of convergence of a power series. This is typically found by taking the absolute value of the terms in the series and applying either a ratio test or a root test. The optimal concept map has nodes and arrows representing 'power series' and ways of determining convergence of power series.

Now that the optimal concept map has been generated and explanations have been given for the nodes in the map, the next section describes phase 1 of the study.

Phase 1 – Student Exam Data

Data collection for phase one was completed as part of a pilot study that occurred during the spring 2015 semester. The participants in phase one were second semester calculus students at a large research university in the northeastern part of the United States. As explained in the introduction, second semester calculus students were chosen because of the researcher's interest in the difficulties students were having with sequences and series, and the literature on student understanding of sequences and series in second semester calculus courses is thin. This particular university was chosen for the researcher's convenience.

There were 53 participants in phase one. This number was chosen because this was the maximum number of exams that could be photocopied before returning the exams to students, and the exams needed to be returned to students in a timely manner. In other words, the researcher had a limited amount of time to photocopy the student work before returning it to them. Consequently, many of the students that participated in phase one were students in the researcher's recitation sections. Many errors were revealed during preliminary analysis of the pilot study data. At the same time, the pilot data also appeared close to saturation in that many students were making similar mistakes. Consequently, it seemed unlikely that collecting exam data from more participants would reveal many new misconceptions.

Data collection in phase one consisted of student solutions to problems on an examination that covered sequences and series, and was collected to address the first research question, “What misconceptions of sequences and series are revealed when students solve problems on sequences and series typically seen in a second semester calculus course?” This examination is typically the last midterm in a second semester calculus course at this university. Exam problems are indicative of the types of problems instructors and professors think that students need to be able to solve, and so it made sense to collect student work to see what mistakes students made, and hence potentially what misconceptions they may hold. In addition, by the time students are tested, they are expected to achieve mastery of the concepts and procedures, while in HW and quizzes, it is understood that students are still developing their knowledge.

The goal of the analysis in phase one was to determine if there was a relationship between student errors and the optimal concept map. Following grounded theory, analysis began with open coding. Open coding was done on a line by line basis, using an initial written description of the student work. The open categories that emerged from the open coding were specific descriptions of student errors on a problem by problem basis. Following open coding was axial coding, in which axial categories were developed that described multiple open categories. Finally, in selective coding, the axial categories were compared to the optimal concept map.

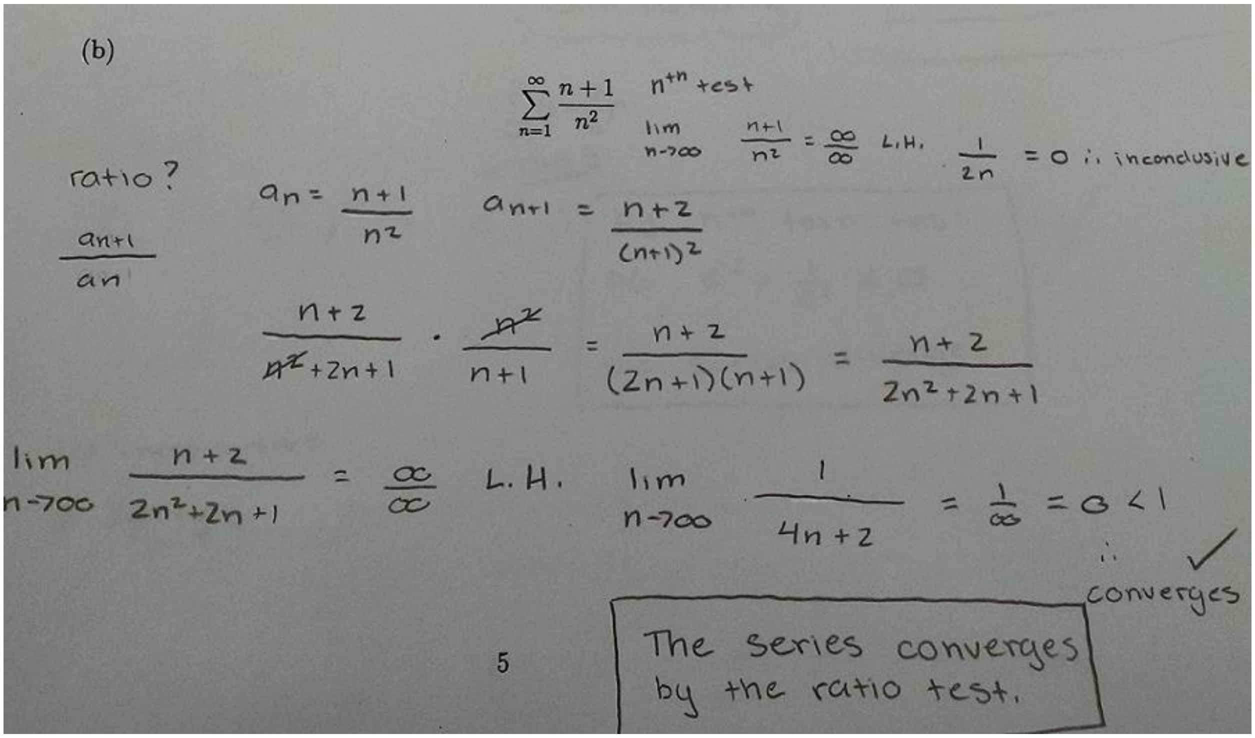
As an example of the coding process, consider problem number 4(b) from the midterm examination. This problem was chosen to analyze first because of the many mistakes students made, such as algebraic manipulation errors and choosing the wrong convergence test, and it is a good example for showing how axial categories were developed. The complete midterm exam can be found in Appendix C. The exam consisted of six questions, some with multiple parts. The first question asked about convergence of sequences, and the other five asked about convergence of series.

The process for analyzing problem 4(b) began with a written description of each student's work. This initial description was coded line by line, looking for anywhere that a student made an error or proceeded correctly. Each error or correct step became an open category. Once every student's solution had been looked at, the researcher had a list of open categories. With the help of another mathematics education graduate student, the researcher began the process of axial coding. We grouped certain open categories together to form new axial categories. Finally, in selective coding, the researcher looked at how each of these axial categories fit or did not fit into a node on the optimal concept map.

An example of this coding process is given below in Figure 3.1. Nine examples of student work are presented, with the initial descriptions as well as the open categories. These examples were chosen because each ultimately gave rise to an axial category.

Figure 3.1: Example of an Algebraic Simplification Error

Student Work, Initial Description, and Categories from Open Coding

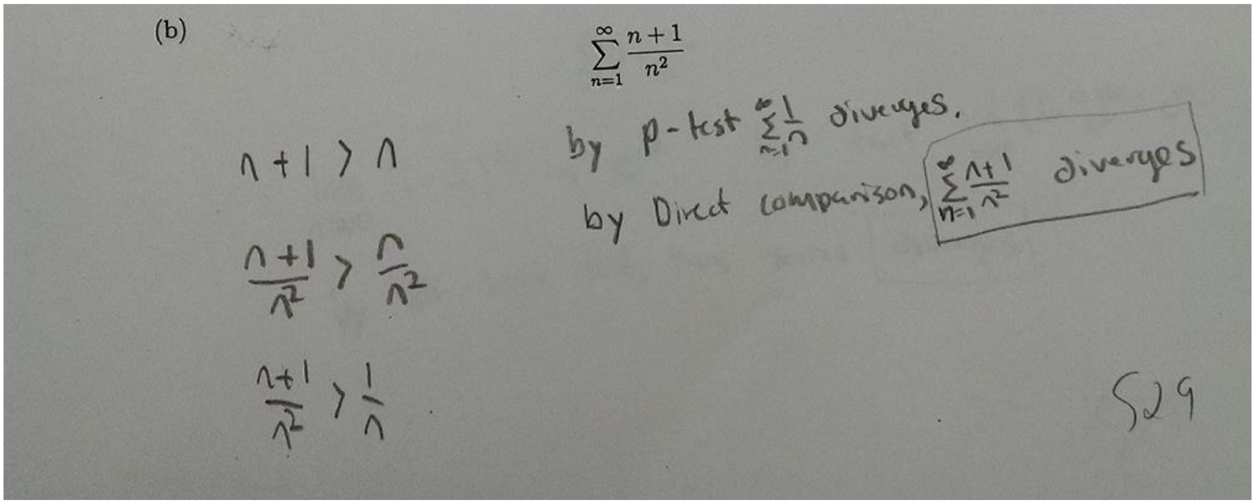


Initial Description: This student correctly realized the nth term test was inconclusive. He then proceeded to try the ratio test. However, in simplifying, the student incorrectly “cancelled” through a sum when he cancelled n^2 terms. Consequently, he ended up getting a value of 0 when taking the limit instead of 1. The ratio test should be inconclusive in this problem, but the algebraic simplification error led the student to say that the series converged by the ratio test.

Categories from Open Coding: nth term test inconclusive, ratio test, incorrect cancelling, algebraic simplification error, series converges by ratio test

Figure 3.2: Example of Correct Work

Student Work, Initial Description, and Categories from Open Coding

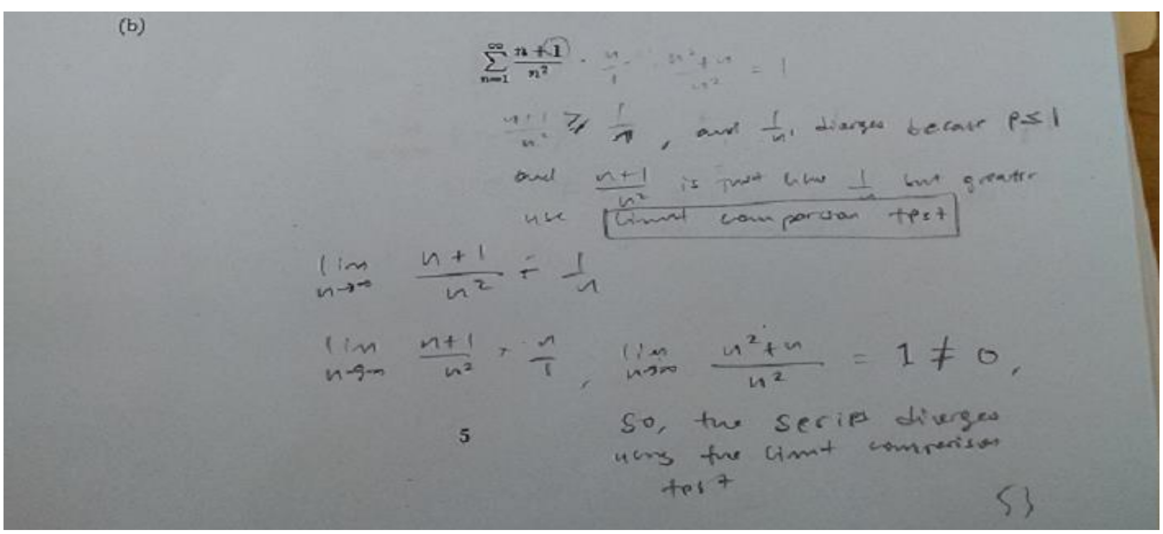


Initial Description: The student started off with a correct comparison, that $n + 1$ is larger than n. Dividing both sides by n^2, the student correctly ascertains that $\frac{n+1}{n^2} > \frac{1}{n}$. He knows that $\sum \frac{1}{n}$ diverges by p-test, and hence correctly concludes by direct comparison that the series in question diverges.

Categories from Open Coding: p-test, direct comparison

Figure 3.3: Example of Work with a Notational Error

Student Work, Initial Description, and Categories from Open Codes

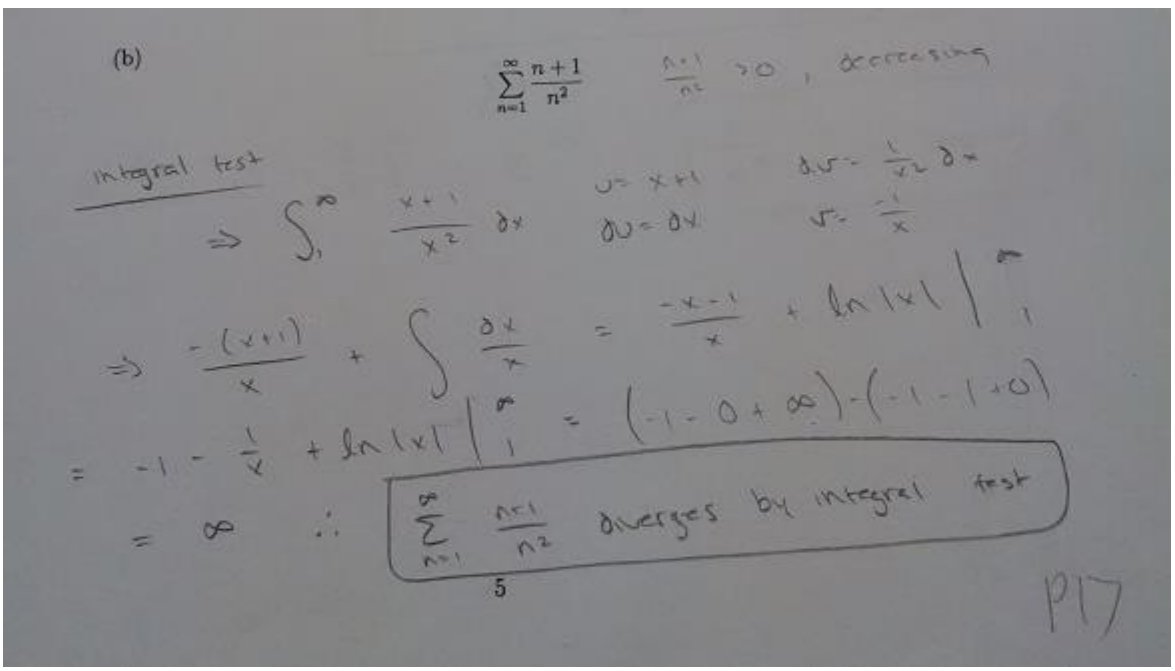


Initial Description: The student correctly states that $\frac{n+1}{n^2} > \frac{1}{n}$. However, the student then indicates that $\frac{1}{n}$ diverges, rather than stating that $\sum \frac{1}{n}$ diverges because the power, p is less than or equal to 1. In other words, the student is missing the series sign in front of $\frac{1}{n}$ when he states that $\frac{1}{n}$ diverges. The student then correctly uses the limit comparison test and reaches the correct conclusion that the series in question diverges by this test.

Categories from Open Coding: limit comparison test, p-test, missing series sign

Figure 3.4: Example of Work with Plugging in Infinity and Not Checking Test Assumptions

Student Work, Initial Description, and Categories from Open Coding

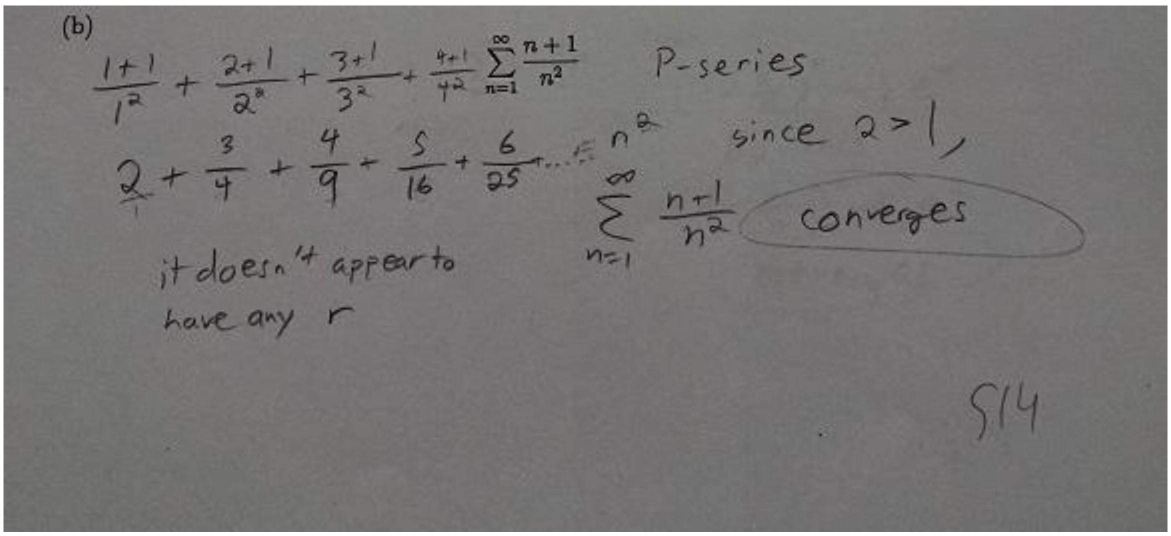


Initial Description: The student indicates that he will use the integral test to solve this problem. However, though the student states the function is decreasing and the terms are positive, he neglects to check that the function is continuous on the appropriate interval. The student correctly uses integration by parts to find the necessary antiderivative, but rather than take limits of the improper integral, the student can be seen “plugging in” infinity. The student ultimately correctly determines that the series diverges.

Categories from Open Coding: Failure to check the assumptions in the integral test, function continuity, integration by parts, antiderivatives, improper integral, plugging in infinity

Figure 3.5: Example Where a Student Chooses an Incorrect Test

Student Work, Initial Description, and Categories from Open Coding

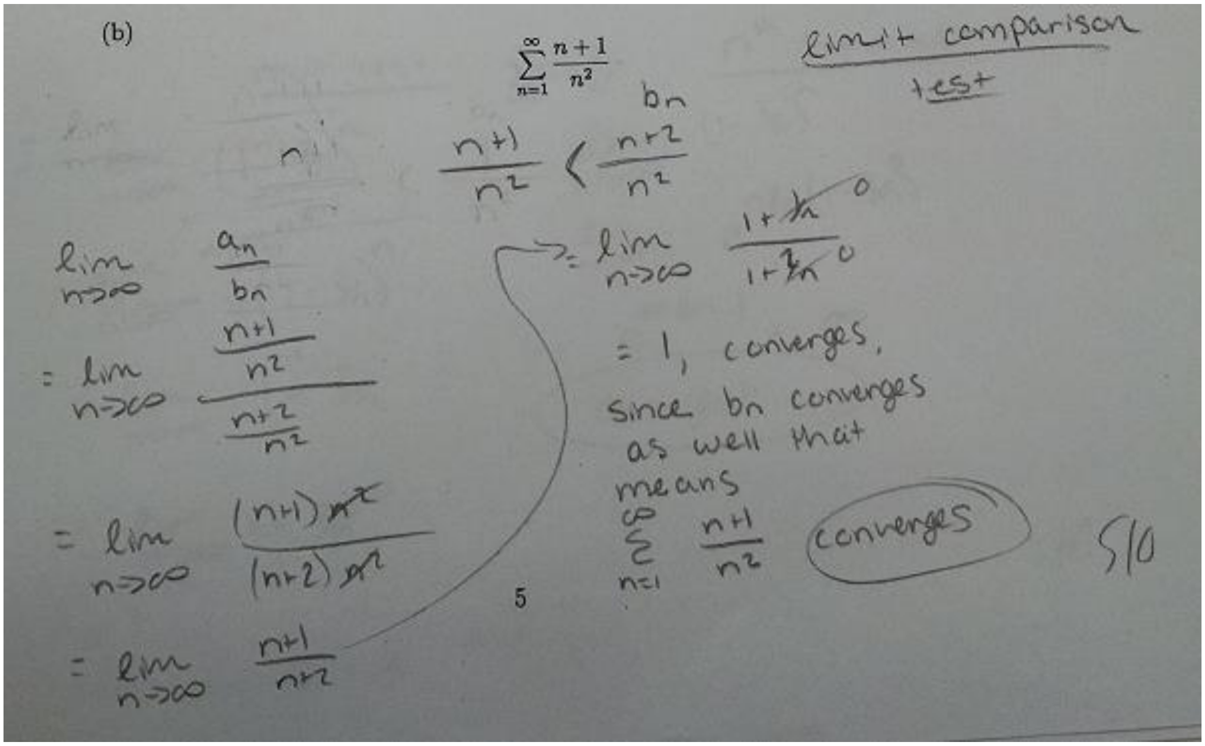


Initial Description: The student writes p-series next to the series in question, even though $\sum \frac{n+1}{n^2}$ is not a p-series. He writes out the first few terms in the series, and then draws the conclusion that the series does not appear to have an ‘r’. This indicates the student might be thinking that this series is a geometric series. He then notes that because 2 is greater than 1, the series in question converges.

Categories from Open Coding: incorrect identification of p-series, incorrect identification of a geometric series

Figure 3.6: Example of Student Work Choosing an Incorrect Function for Comparison

Student Work, Initial Description, and Categories from Open Coding

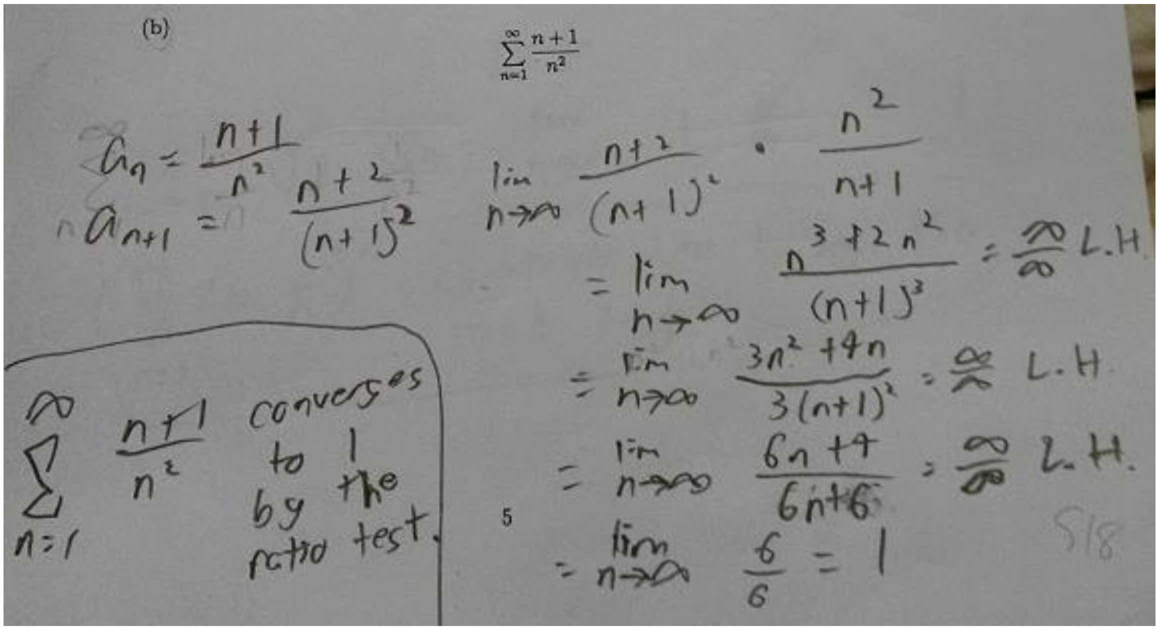


Initial Description: The student correctly states that $\frac{n+1}{n^2} < \frac{n+2}{n^2}$. He then uses the limit comparison test and correctly gets a limit of 1. However, he concludes that the series in question converges because $\sum \frac{n+2}{n^2}$ converges, which it does not. It is unclear how $\frac{n+2}{n^2}$ would help the student, as the behavior is unknown without a comparison to something well known such as $\frac{1}{n}$.

Categories from Open Coding: limit comparison test, choice of function

Figure 3.7: Example of Student Work Using Wrong Test and Drawing False Conclusion

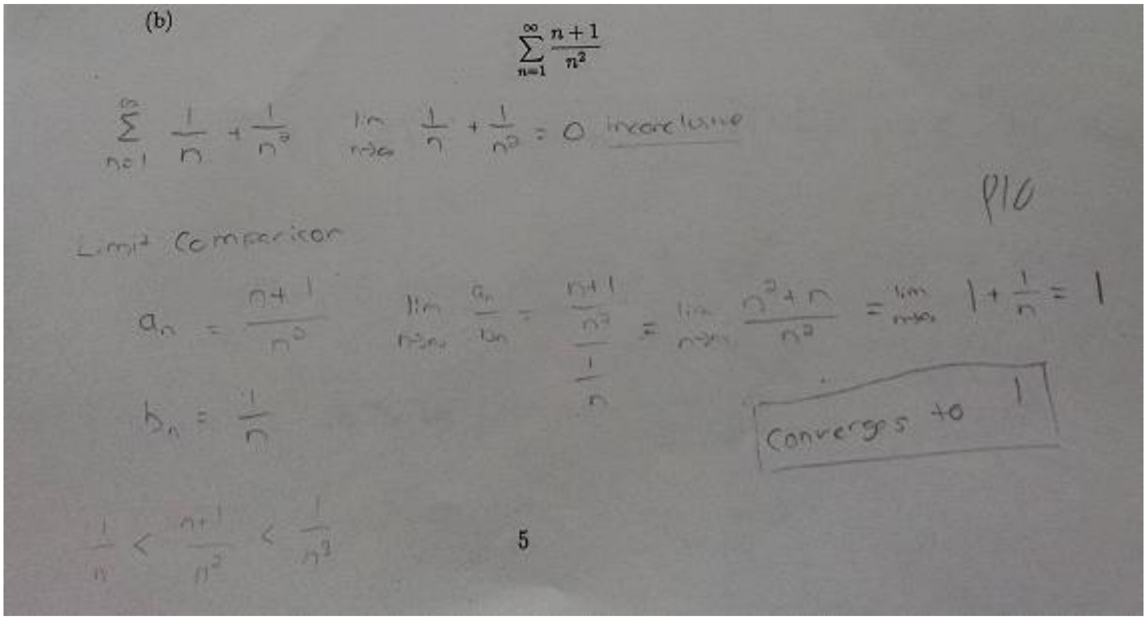
Student Work, Initial Description, and Categories from Open Coding



Initial Description: Student tries to use the ratio test to solve this problem. He evaluates the limit of the ratio correct, obtaining a value of 1. However, this should indicate that the test is inconclusive. The student instead concludes that the series converges to the value of the limit by the ratio test.

Categories from Open Coding: ratio test, limit value of 1, series converges

Figure 3.8: Example of Student Work Using Correct Test, but Drawing False Conclusion
Student Work, Initial Description, Categories from Open Coding



Initial Description: Student correctly realizes the nth term test is inconclusive. The student proceeds to use a limit comparison with $\frac{1}{n}$, and correctly obtains a limit of 1. However, rather than state that this limit tells us the series in question diverges, he says that the series converges to 1.

Categories from Open Coding: limit comparison, limit of 1 means series converges to 1

Once open coding was complete, a list of all the open categories was compiled, and with the help of a mathematics education graduate student that had experience as an instructor for second semester calculus, the researcher looked for ways to combine these categories. The following table, figure 2, lists all categories that developed as a result of axial coding of problem 4(b) from the pilot study data. These axial categories were first presented in Earls and Demeke (in press), though No Mistakes was renamed to Correct Work to better represent the open categories it describes. For example, the open category “nth term test inconclusive” was correct work, even though the student that stated this fact in figure 1.1 above made mistakes on the problem.

Figure 3.9: Axial Categories, Abbreviations, and Explanation Table for Series
Axial Categories, Abbreviations, and a Brief Explanation with an Example for Series

Category	Abbreviation	Explanation
Correct Work	CW	Student work following a correct approach. CW included for this problem using a comparison test or an integral test.
Notational Error	NE	A notational error. For example, a student says $\frac{1}{n}$ diverges without including the series symbol.
Algebra	A	An algebraic error. A student might, for example, "plug in" infinity, or incorrectly simplify a rational expression by "cancelling" through a sum
Function Choice	FC	Wrong function choice when using a comparison test. For example, a student might try to make a comparison with $\frac{1}{n^2}$.
Unchecked Assumptions	UA	Student failed to check that the function satisfied the assumptions in the integral test.
Algebra error leading to Incorrect Test Choice	AITC	Student reaches a false conclusion (usually in the ratio test) because of an algebraic mistake. This mistake typically was cancelling through a sum.
Incorrect Test Choice	ITC	Student chooses an incorrect test, such as an nth term test, or a geometric test.
Wrong Conclusion Drawn from Test	WCDT	Student reaches an incorrect conclusion when using a test. For example, a student uses the ratio test and says that a value of 1 means the series converges.

Each of these axial categories was created by merging one or more open categories together. For example, the category Correct Work (CW) encompasses the open categories "nth term test inconclusive", "p-test", "direct comparison" and "limit comparison test." This is because using a p-test along with a direct comparison or limit comparison test is the correct way to solve this problem. Identifying the nth term test as inconclusive is also correct, even if it doesn’t immediately lead to a solution to the problem. As another example, Incorrect Test Choice (ITC) was formed by merging open categories such as “ratio test”, “incorrect identification of p-series”, and “incorrect identification of geometric series.” The ratio test cannot be used in this problem because it is inconclusive, and while a p-test can help in conjunction with a comparison test, the series in question is not a p-series or a geometric series. Trying to use any of these three tests in this situation is choosing an incorrect test.

The final stage of analysis in grounded theory is selective coding, the purpose of which is to generate the theory on student misconceptions in second semester calculus courses. As mentioned in the section above on grounded theory, in this study a theory was developed but it might be incomplete. Hence, the purpose of selective coding in this study was to see how the axial categories related to this developed theory on misconceptions.

The optimal concept map is a visual representation of the theory on student misconceptions. Thus, in selective coding, the researcher analyzed how each of the categories developed during axial coding “fits” into the optimal concept map, and indicated any places where the optimal concept map failed to consider a particular axial category. An example is given below using the axial categories that developed from the analysis of problem 4(b) in the pilot study.

The first axial category that developed was Correct Work (CW). CW encompassed such open codes as comparison tests, integral tests, and p-series. The optimal concept map indicates students should have an understanding of comparison tests and integral tests. However, the optimal concept map originally had no nodes referring to p-series tests. Since this is part of CW, the optimal concept map need was adjusted to account for student understanding of p-series.

The second axial category was Notational Errors (NE). NE does not have an explicit node in the optimal concept map. Hence, the optimal concept map should be updated to include a proper understanding of notation used when referring to sequences and series.

The third axial category was Algebra (A). Such an algebraic error included plugging in infinity, rather than taking a limit. But this error occurred as a result of trying to evaluate an improper integral, which is part of the integral test. Hence, the integral test node accounts for such algebraic understanding.

The fourth axial category was Function Choice (FC). Choosing a function to compare to is a critical part of using comparison tests. Comparison tests, both direct and limit, are nodes in the optimal concept map.

The fifth axial category was Unchecked Assumptions (UA). Knowing what the assumptions are for the integral test, and making sure these assumptions hold for the problem in question, is a part of having an understanding of the integral test. Integral test is a node in the optimal concept map.

The sixth axial category was Algebra error leading to Incorrect Test Choice (AITC). There are two issues here. The first is the algebra error. This algebra error involved the incorrect simplification of a rational function. But a series is a type of sequence which is a type of function. Rational functions are included in the expert concept maps for the function concept. Consequently, the algebraic understanding necessary here is encompassed by the function concept. Function is a node in the optimal concept map. In particular, student knowledge of simplification of rational functions would have helped students avoid this error.

The second issue is the Incorrect Test Choice (ITC), which was also the seventh axial category. While the algebraic mistake did lead to the incorrect test choice, students who made errors categorized by AITC still worked with a test that could not have been used to successfully solve this problem. Students who made errors categorized by ITC did not make an algebra error, but still worked with a test that would not have helped solve the problem. Test selection and test choice are both nodes in the optimal concept map.

The eighth axial category was Wrong Conclusion Drawn from Test (WCDT). Regardless of which test a student chooses, he needs to know what the conclusion of that test says. Each test has a node in the optimal concept map, and having an understanding of a test includes knowing the conclusion of the test.

Based on selective coding, it appeared that the existing theory had not addressed issues of p-series. An Additional node was added to the optimal concept map to encompass these changes.

Phase 1 – Summary. The purpose of the data collected in phase 1 was to address the first research question, "what misconceptions of sequences and series are revealed when students solve problems on sequences and series typically seen in a second semester calculus course?" Data collected were student responses to an exam on sequences and series given in a second semester calculus course. Data analysis closely followed grounded theory; open coding was used, and categories emerged during open coding. Axial coding was used to develop categories that encompass the categories that emerged during open coding. Finally, selective coding was used to determine what is missing from the existing theory.

Phase 2 – Task-Based Interviews

The purpose of this phase is to address the second and third research questions. By having students discuss their solutions, some insight can be gained into their thought process and why they might be making certain errors.

The total number of participants for task-based interviews was twelve. Seven participated in a pilot of the interviews, while five participated in interviews for the full study. After analyzing all twelve interviews, it appears the data was saturated. In other words, similar mistakes were appearing amongst interviewees. Hence, little new data would be gained by interviewing more students.

Data collection in phase two consisted of semi-structured task based interviews, (Goldin, 2000). Students were given mathematical tasks to solve involving sequences and series, and they were asked to explain their reasoning as they worked through the problems. Clarifying questions were asked whenever a student explanation was incomplete or unclear. The purpose of these interviews was to provide further data to answer the first research question as well as to answer the second and third research questions, “In what ways, if at all, do the misconceptions revealed relate to the prerequisite knowledge students are expected to have prior to starting a second semester calculus course” and “What additional knowledge or conceptualization of sequences and series do students need to be successful in second semester calculus courses?”

The mathematical tasks that the students were asked to solve can be found in Appendix D. To ensure that these problems were typical for a second semester calculus course, they were designed to be similar to the exam problems from phase one. All tasks were confirmed typical by an experienced second semester calculus instructor.

A list of the tasks, the exam problem they relate to, and how they related to the optimal concept map if solved correctly are given in table 3 below. If the task does not relate to an exam problem, this is also indicated in the table. Note that some nodes in the optimal concept map were not covered in the interviews due to time constraints. These nodes that students were not interviewed about were divergent, showing a limit does not exist, bounded and monotonic, sequence of partial sums converges, and telescoping series.

Table 3.11: Interview Tasks, Phase One Exam Problems, and Nodes in Optimal Concept Map

Interview Task, the Exam Problem It Is Similar To, and How it Relates to the Optimal Concept Map

Interview Task	Phase One Exam Problem	Node(s) in Optimal Concept Map
Suppose you know that $\lim_{n\to\infty} a_n = 3$. What can you say about: $\sum_{n=0}^{\infty} a_n$ Suppose you know that $\sum_{n=0}^{\infty} b_n = 3$ What can you say about $\lim_{n\to\infty} b_n$	None: Recommended by an experienced second semester calculus instructor	Series, diverge, test selection, nth term test, sequence, function, convergent, finding limit
Explain your reasoning as you determine whether the series converges or diverges. If it converges, explain how you would go about finding the sum, or explain why you cannot find the sum. $\sum_{n=1}^{\infty} \pi^n e^{-2n}$	Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work. $\sum_{n=0}^{\infty} e^{-3n}$	Series, converge, geometric series $\lvert r\rvert < 1$, root test, ratio test, test choice

Explain your reasoning as you determine whether the series converges or diverges. If it converges, explain how you would go about finding the sum, or explain why you cannot find the sum. $$\sum_{\boldsymbol{n=1}}^{\infty} \frac{\boldsymbol{2n+1}}{\boldsymbol{(n+1)^2}}$$	Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work. $$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$	Series, diverge, test selection, limit comparison/direct comparison test/integral test
Explain your reasoning as you determine whether the series converges or diverges. If it converges, explain how you would go about finding the sum, or explain why you cannot find the sum. $$\sum_{\boldsymbol{n=1}}^{\infty} \frac{\boldsymbol{(-4)^n}}{\boldsymbol{(2n+1)}!}$$	Determine whether the following series converges absolutely, converges conditionally, or diverges. Give reasons for your answer, including any test you may have used. $$\sum_{n=0}^{\infty} \frac{(-3)^n}{(n+1)!}$$	Series, converge, absolutely, test choice, ratio test
Explain your reasoning as you find the values of *x* for which the following power series converges and the radius of convergence of the series. $$\sum_{\boldsymbol{n=1}}^{\infty} \boldsymbol{(-1)^{n-1}} \frac{\boldsymbol{x^n}}{\boldsymbol{n}}$$	None: Confirmed typical by an experienced second semester calculus instructor.	Series, power series, radius of convergence, root test/ratio test, converge, conditionally, alternating series test, diverge, test selection, p-series test
Suppose that $\boldsymbol{a_n = \frac{\sin\frac{(2n-1)\pi}{2}}{n}}$. What can you say about $\boldsymbol{\lim_{n\to\infty} a_n}$? What can you say about: $$\sum_{\boldsymbol{n=1}}^{\infty} \boldsymbol{a_n}$$	Determine whether the following sequences converge or diverge. If the sequence diverges, specify whether it diverges to ∞ or −∞ if that is the case. Find the limit of all convergent sequences. Justify your response by showing your work. $$a_n = \frac{\cos(n\pi)}{\pi}$$	Sequence, convergent, function, sandwich theorem, series, converge, conditionally, alternating series test

	Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+1}$	

Interviews were video recorded, with the camera focused on student work. Interviews were carefully transcribed for data analysis. Data analysis followed the grounded theory analysis described earlier. Open coding was done on a line by line basis. Categories emerged during open coding, and axial coding was used to develop connections between the categories. Finally, selective coding was used to relate the axial categories to the optimal concept map.

Two examples of this analysis process from the pilot study interview transcripts are given below. These examples were chosen because they show how the interview data was used to answer the second and third research questions. In particular, the first excerpt reveals a misconception that seems unrelated to prerequisite knowledge, while the second excerpt reveals a misconception related to the function concept, which is related to prerequisite knowledge. Line numbers have been added to the transcripts for reference in the open coding. The first excerpt is from the student Amanda, and is from the first task in the interview. The second excerpt is from the student Scott, and is from the second task in the interview. Note that pseudonyms are being used for each student.

The two examples below show the line by line process for open coding, as well as the categories that developed during that open coding. The axial categories encompass more open categories than what is seen in these examples.

(1)Amanda: Um, ok, so I see that the limit as n approaches infinity a_n goes to 3, so the sum a_n, first I set
(2)up, um, the limit, as approaches infinity of a_n. That's how I would try to solve, um, for the series.

(3)Interviewer: Ok.

(4)Amanda: That goes to, I know that that goes to 3. Suppose that (inaudible) equals 3. Are, um…hmm.
(5)I think I could, I think I could assume that the, that this limit (points to limit of b_n)…the sum is equal to 3.

(6)Interviewer: Sorry, those are two…

(7)Amanda: And the limit…

(8)Interviewer: These are two separate questions.

(9)Amanda: Yeah, oh, ok. Oh, I see. Ok…So that's 3 (points to line 1 of her work, $\lim_{n\to\infty} a_n = 3$). And the
(10)sum…I'd say the sum goes to three 'cause as it gets bigger (points to the same thing)…the, as it gets
(11)bigger it seems to just go to a value.

(12)Interviewer: As what gets? Sorry, you said "it".

(13)Amanda: Um, Sorry.

(14)Interviewer: What is "it"?

(15)Amanda: As the limit, as a_n goes to infinity.

(16)Interviewer: Ok.

(17)Amanda: It seems to be approaching a value.

(18)Interviewer: Ok.

(19)Amanda: So I assume that the series also would go to that value.

(20)Interviewer: Ok.

(21)Amanda: Um, then the second one, I would have to say that, uh, the limit for the second one, oop, n
(22)goes to infinity, b_n, would therefore…say that'd go to the value as n approaches infinity.

(23)Interviewer: Can you explain why?

(24)Amanda: Um, I don't…it actually, I think it would actually go to infinity because if every n value it (25)seems to go to 3, so when you add those up as it goes to infinity, you would just get very large.

(26)Interviewer: Ok.

(27)Amanda: So…Is this good if I wrote that?

(28)Interviewer: Mmhmm, that's fine.

(29)Amanda: Ok.

Open Coding: In line (2), the student tries to find the series sum by setting up the limit of the sequence. In line (5), Amanda points to the limit of a sequence and refers to it as a sum. In line (10), Amanda says that the sum goes to 3 because as "it" gets larger, the sum seems to go to a value. She confirms in line (15) that "it" refers to the limit as a-n goes to infinity. In lines (17) and (19) she argues that the sum would go to the value of 3 because it should go to the same value as the limit of the sequence. In lines (21) and (22) she argues that b-n would go to the value (of 3) as n approaches infinity. When asked to explain why in line (23), she changes her mind and says that the sequence goes to infinity, because when you repeatedly add 3, it would get very large.

Open Categories: Thinking of a series as a sequence, thinking of a sequence as a series

Axial Code: Confusion between sequences and series (merges both open categories above)

(1)Scott: So this would equal (series given in problem) pi to the n times e to the negative 2 times e to the (2)n.

(3)Interviewer: Ok.

(4)Scott: Because, you can, you can just pull out the pow, power. And since e, since 1 over e squared is a (5)constant you can pull it out in front. Um, you get pi to the n times e to the n. And this would (6)equal…oh. Pi times e to the n. And then you can do root test here.

(7)Interviewer: Ok.

(8)Scott: To figure out it's convergence. Um, or, oh no cause that's geometric. No. No it's not…I think I (9)can make it geometric. If I pull out a, so I need to make this sum be from n equals 1 for it to be geo, (10)err, n equals 0 for it to be geometric and have it be raised to the, still raised to the n.

(11)Interviewer: Ok.

(12)Scott: So I'd wanna multiply it by, cause we're going backwards in it (the sum) so you'd want, so it, it (13)would be plus 1, so it, just times it by pi times e. Oh, but then that wouldn't , uh, yeah that would be (14)constant. So I pull that out in front, too. Which equals, which equals pi, pi over e, n equals 0, pi (15)times e to the n. And, so this, could do the geometric, uh, series, I don't know if it's called the (16)geometric series test, but uh, this is your r (the pi times e). And the absolute value of r is greater (17)than 1 so it diverges.

(18)Interviewer: Ok. I don't have any questions (laughs).

(19)Scott: Ok. (laughs)

Open Coding: In lines (1)-(2), the student incorrectly simplifies e^{-2n} as $e^{-2} * e^{n}$. He justifies this in line (4) by saying you can just pull out the power. In lines (5)-(6) he pulls the e^{-2} in front of the summation sign, which leaves him inside the sum with $(\pi * e)^{n}$, and initially indicates he would use the root test to determine convergence. In line (8) he identifies this as a geometric series, though because the index starts with a 1, he needs to pull out some terms to make the index start with a 0 instead as indicated in lines (9) – (10). He outlines this process in lines (12)-(15). In lines (16)-(17), he notes that the r value for the geometric series is greater than 1, and thus concludes that the series diverges.

Open Categories: Incorrect simplification of an exponential function, root test to determine convergence, identification of geometric series, re-indexing

Axial Categories: Algebra of exponential functions, Correct Work

In terms of selective coding, all axial codes in these examples fit into the optimal concept map. Confusion between sequences and series is handled by the fact that there are two separate nodes, one for sequences and one for series. In particular, the optimal concept map indicates that students should understand the difference between a sequence and a series. The second axial category, Algebra of Exponential Functions, fits into the function node in the optimal concept map. Correct Work involved identifying the series in question as a geometric series and re-indexing.

Phase 3 – Multiple Choice Test

The purpose of phase 3 was to see how the developed theory on student misconceptions of sequences and series fits into a larger population. Creswell (2013) notes that grounded theory studies sometimes have a large scale quantitative component. Data collection in phase three served this purpose, and hence also provided more data for answering the three research questions stated in the introductory chapter.

In phase three, 185 students were given a multiple-choice test with questions about sequences and series. A multiple-choice test format was chosen because of the large number of students that were given the assessment, and multiple choice tests are used in many large-scale assessments, such as the Scholastic Aptitude Test (SAT), Advanced Placement (AP) exams (The College Board, n.d.), and international exams such as the Programme for International Student Assessment (PISA) exam (Organization for Economic Co-operation and Development, n.d.). This study was conducted in the spring 2016 semester, and 185 is the number of students registered for second semester calculus at this research university that were willing to take the assessment.

Because the researcher was interested in misconceptions that arose as students solve typical second semester calculus problems, it was important that these multiple-choice questions were similar to problems that students would see in their course. Consequently, the multiple-choice items were the same as those that were asked in interviews in phase two of this study. These questions were already reviewed by an experienced second semester calculus instructor, which helped validate the appropriateness of the questions.

Kehoe (1995) recommends three to four well written choices for each multiple-choice item. Distractor answers were chosen by looking at pilot study data from interviews. Incorrect responses given by students during the interviews were used as distractors. When interview data did not provide enough distractors, pilot study exam data from phase one was used. Ultimately, since some questions had fewer incorrect responses than others, a total of three distractors were given for each question, meaning each question had four possible answer choices.

In addition to collecting student responses to the multiple-choice items listed above, demographic information was collected from students taking this examination. The purpose of the cover sheet was to obtain information that might be useful in data analysis. For example, the first question on the cover sheet asks for information about students' prior mathematics courses. By knowing about a student's previous mathematics courses, some insight was gained on their preparation for second semester calculus and their prerequisite knowledge. Having this information provided an opportunity to determine if there was any connection between students that do well on this multiple-choice item and the types of courses they took prior to entering second semester calculus. The full cover sheet can be found in Appendix F. The rationale behind asking for the rest of the information on the cover sheet is discussed below.

The second question asks if students had seen sequences and series prior to entering second semester calculus courses. Exposure to sequences and series prior to entering the course may have influenced student understanding of sequences and series.

Age, year, gender, and race could all have implications for future research, and were collected to provide a rich description of the sample. The last question about anticipated grade in the course was useful for determining if there was a correlation between success in the course and success on the multiple-choice items. A correlation between high grades and high scores would help to further emphasize that the problems being asked are indeed typical problems.

There are many different options for scoring multiple choice tests. Traditional scoring involves assigning a value of one for each correct answer, and a value of zero for each incorrect answer. However, Abu-Sayf (1979) has criticized this method for encouraging guessing, even rewarding students when they guess correctly. Other scoring methods include a formula that corrects for guessing and awarding credit for partial knowledge (Kurz, 1999).

One way of awarding credit for partial knowledge is to use “option weighting”. In option weighting, each answer choice is given a different value depending on how close that choice is to the correct answer (Crocker & Algina, 1986). In other words, answer choices can be ranked based on level of correctness. Each answer choice is ranked from a 0 (representing totally incorrect) to a 1 (representing totally correct). For example, suppose a student were given a series that converged, and asked to determine if the series converged or diverged. A student that knows the series converges but for an incorrect reason would receive more credit than a student that says the series diverges.

Option weighting was chosen because each answer on the multiple-choice test is indicative of different types of student thinking observed during interviews. The multiple-choice test is presented below, with an indication of the number of points awarded for each answer choice as well as notes for the source of each distractor. A brief rationale is provided for each answer choice. The test was reviewed by three experts in the field, and some wording was changed as a result of their feedback. In addition, an experienced mathematics educator with a strong background in statistics recommended that the most difficult problem be moved to the end, and so the first interview task, which was the most difficult interview problem, is the last multiple choice item. The complete multiple choice test without explanations for the distractors can be found in Appendix E. A four-sided die was rolled to determine the position of the correct answer (similar to Kehoe’s (1995) recommendation of using coin flips). Note that since the multiple-choice items are the same as the interview questions, the same nodes in the optimal concept map were addressed by these questions.

1. The series
$$\sum_{n=1}^{\infty} \pi^n e^{-2n}$$
   A. Converges by the nth term test for divergence – 0 (totally incorrect)
      Interview (Shawn)
   B. Is a divergent geometric series – ½ (partially correct, it is geometric)
      Interview (Scott)
   C. Is a convergent geometric series – 1 (totally correct)
      Correct Answer
   D. Is convergent by the root test, but is not a geometric series – ½ (partially correct, it is convergent)
      Interview (Zoey and Becca)

2. What can you say about the series:
$$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$
   A. Diverges by a comparison test – 1 (totally correct)
      Correct Answer
   B. Converges by the integral test – ½ (partially correct – integral test could be used with partial fractions, but the series diverges)
      Interview (Amanda)
   C. Converges by a comparison test – ½ (partially correct in identifying the need for a comparison test, but incorrect execution)
      Interview (Katherine)
   D. Both the ratio and nth term tests are inconclusive, so we cannot say whether this series converges or diverges. – 0 (totally incorrect – inconclusive tests mean we need to try something else)
      Interview (Becca) – though she did continue on to get the answer correct, I chose this answer in case a student might stop at inconclusive

3. The series
$$\sum_{n=1}^{\infty} \frac{(-4)^n}{(2n+1)!}$$
   A. Converges by the alternating series test, but is not absolutely convergent – ½ (partially correct, it does converge, but it does converge absolutely)

Interview (Henry) – though there was more going on than just this – also from Exam (S18)

B. Converges absolutely – 1 (totally correct)
Correct Answer
C. Diverges by limit comparison – 0 (totally incorrect – the series converges, and limit comparison does not help here)
Exam (S11)
D. Diverges absolutely – 0 (totally incorrect – series converges absolutely)
Exam (P7, problem 5)

4. What is the interval of convergence for the following power series?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

A. $-1 < x \leq 1$ – 1 (totally correct)
Correct Answer
B. $0 < x \leq 1$ – ½ (partially correct, is within the interval of convergence)
Interview (Henry)
C. Convergent for all x – 0 (totally incorrect, clearly not true for large values of x)
Interview (Katherine)
D. Divergent for all values of x – (totally incorrect, series clearly converges for at least x=0)
Interview (Shawn)

5. Suppose that $a_n = \frac{\sin\frac{(2n-1)\pi}{2}}{n}$. What can you say about $\lim_{n\to\infty} a_n$?

A. Alternates from -1 to 1 forever – ½ (partially correct, describes the behavior of sin)
Interview (Henry) – final answer
B. $a_n \geq \frac{-1}{n}$, so by the comparison test, $\lim_{n\to\infty} a_n$ diverges. – ½ (partially incorrect, the inequality is correct, but comparison tests are used for series, not sequences)
Interview (Katherine)
C. Using L'Hôpital's rule, we have $\lim_{n\to\infty} a_n = \infty$ - 0 (totally incorrect, assumptions for L'Hôpital's rule are not satisfied)
Interview (Henry) – first answer
D. $\lim_{n\to\infty} a_n = 0$ by the Sandwich Theorem – 1 (totally correct)
Correct Answer

6. Suppose that you know that

$$\sum_{n=1}^{\infty} b_n = 3$$

What can you say about $\lim_{n\to\infty} b_n$?

A. $\lim_{n\to\infty} b_n = \infty$ - 0 (totally incorrect, sequence must converge to 0 since series converges)
Interview (Amanda)

B. $\lim_{n\to\infty} b_n = 0$ – 1 (totally correct)
Correct Answer

C. $\lim_{n\to\infty} b_n = 3$ – ½ (partially correct, sequence must converge, but it must converge to 0)
Interview (Becca, Katherine, and Shawn)

D. We can't say anything about $\lim_{n\to\infty} b_n$ – 0 (totally incorrect, sequence must converge and to 0)
Interview (Zoey)

Data analysis proceeded in two phases. The first phase was exploratory data analysis (EDA), and the second phase was confirmatory data analysis (CDA). Behrens (1997) explains that is important to begin with EDA before moving to CDA because EDA allows researchers, "…to find patterns in the data that allow researchers to build rich mental models of the phenomenon being examined" (p. 154).

The process of EDA began with analyzing each variable individually, looking for outliers and inspecting the shapes of the distributions (Curtis, D. & Araki, C., 2003). Box plots were useful for identifying outliers (Behrens, 1997). After looking at the individual variables, EDA continued by analyzing relationships between two variables, and then finally looking at multiple variables at once (Curtis, D. & Araki, C., 2003).

In this study, the individual variables included overall scores and scores on a particular problem from the multiple-choice test, as well as prior math courses, age, gender, year, race, experience with sequences and series, and expected grade in the course. The JMP software was used to aid in exploratory data analysis. The JMP software helped identify mean, median, mode, and variance as well as the shape of the distribution when looking at a single variable. Scatterplots and contingency tables were used to determine if correlations existed when looking at two variables at a time. Bubble plots were used for multivariable analysis.

This EDA informed the next phase, CDA. Though CDA cannot be explicitly described, some hypotheses can be described. For example, one hypothesis was that students that have seen sequences and series in a previous course performed better overall on the multiple-choice test than students that have not. This hypothesis was important because it relates to the third research question. Another was that students whose prior three courses include some level of high school mathematics performed better than those students who took precalculus and first semester calculus at this research university, and this relates to the second research question.

Testing hypotheses is important because, in CDA, researchers state a null hypothesis and an alternative hypothesis and use a statistical significance test to determine whether or not the null hypothesis should be rejected (Curtis, D. & Araki, C., 2003). Choosing a statistical significance test depends on what is discovered during EDA. For example, using a t-test will require a normal distribution.

Errors and Misconceptions

The methodology in phases one through three describe ways of analyzing student errors. However, the first research question addresses student misconceptions. Consequently, it is important to distinguish between errors and misconceptions.

Cangelosi, Madrid, Cooper, Olson, and Hartter (2013) describe the difference between errors and misconceptions in their study about students' difficulties with negative signs in exponential expressions. They noted that errors may not always be a sign of a misconceptions, but persistent errors, errors that students made in precalculus and college algebra and were still being made consistently in first and second semester calculus, were indicative of potential misconceptions.

Other researchers have similarly noted the differences between misconceptions and errors. Hadjidemetriou and Williams (2002) define errors as "erroneous responses to a question" (p. 69), and Smith, diSessa, and Rochelle (1993) define a misconception as a "conception that produces a systematic pattern of error" (p. 119).

Following the work of Cangelosi et al. (2013), the errors described in the results section of this study will be defined as potential misconceptions if these errors are in some way persistent errors. Errors will be identified as persistent in one of two ways. The first way follows Cangelosi et al. (2013) closely; if errors that students made in second semester calculus are the same as errors that are made in precalculus or college algebra, the errors will be identified as persistent. The second way will be to identify errors made in two different semesters: Spring 2014 (exam data and pilot interviews) and Spring 2015 (full interviews and multiple choice assessment). By showing that the errors were made twice in different years, an argument can still be made that the errors are persistent.

Cangelosi et al. (2013) used a large scale quantitative analysis to confirm that errors were being consistently made across multiple courses. In the absence of such data in this study, persistent errors will not be definitively categorized as misconceptions, but rather as potential misconceptions.

Summary

The purpose of this chapter was to describe the methodology behind a study aimed at answering the three research questions stated in the introduction. The study proceeded in three phases. In the first phase, student exam data was collected and analyzed using a grounded theory approach. In the second phase, interview data was collected, and analysis was again guided by grounded theory. The third phase involved a statistical analysis of a multiple-choice examination, starting with exploratory data analysis and moving to confirmatory data analysis.

The results of this study as well as a discussion of these results is presented in the next chapters.

Chapter 3 – Qualitative Analysis and Results

The purpose of this chapter is to discuss the results of the study using the methodology described in chapter two. Because the study is divided into a qualitative component and a quantitative component, the results will be divided into two parts, one for qualitative analysis and one for quantitative analysis. This chapter focuses on the qualitative results.

Student Exam Work and Interviews

The exam data consisted of one question on sequences with three parts, and five questions on series. The interview had 2 questions about sequences, and the rest about series. Because approaches to solving problems are vastly different between sequences and series (for example, the ratio test can be used on series but not sequences), this section is divided into two parts: one for sequences and one for series.

Sequences. Several axial categories arose as a result of analyzing student exam data on sequences. These categories were Notational Error (NE), Algebra (A), L'Hôpital (LH), Incorrect Behavior Argument (IBA), Test Invocation (TI), and Trigonometry (Trig). Table 4.1 lists each of those codes with a brief description. Each of these categories is then discussed further, with examples from the data.

Table 4.1: Axial Categories, Abbreviations, and Explanation Table for Sequences
Axial Categories, Abbreviations, and a Brief Explanation

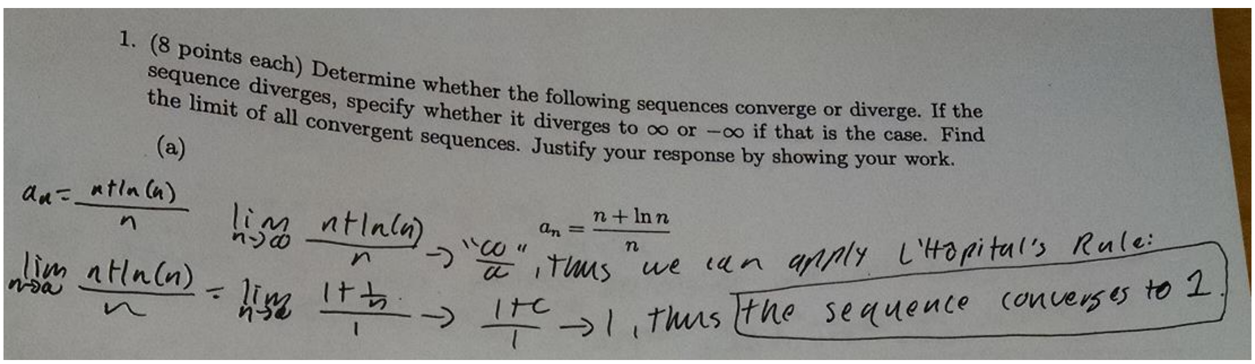
Category	Abbreviation	Explanation
Notational Error	NE	A notational error. For example, a student may have used an arrow when an equals sign should have been used
Algebra	A	An algebraic error. A student might, for example, "plug in" infinity, treating it like a real number.

L'Hôpital	LH	Student chose to use L'Hôpital's rule when it did not apply
Incorrect Behavior Argument	IBA	Student incorrectly indicated the end-behavior of a non-trigonometric function.
Test Invocation	TI	Student used a series test on a sequence problem, or neglected to explicitly state the use of the Sandwich Theorem.
Behavior of Trigonometric Functions	Trig	Student argued that sin or cos was not continuous, or did not know the end behavior of trigonometric functions.

notational error. Notational errors included using an arrow when an equals sign should have been used, or not including the explicit name of the function which the student was taking the limit of. See figure 4.2a and figure 4.2b:

Figure 4.2a: Example of Work with a Notational Error

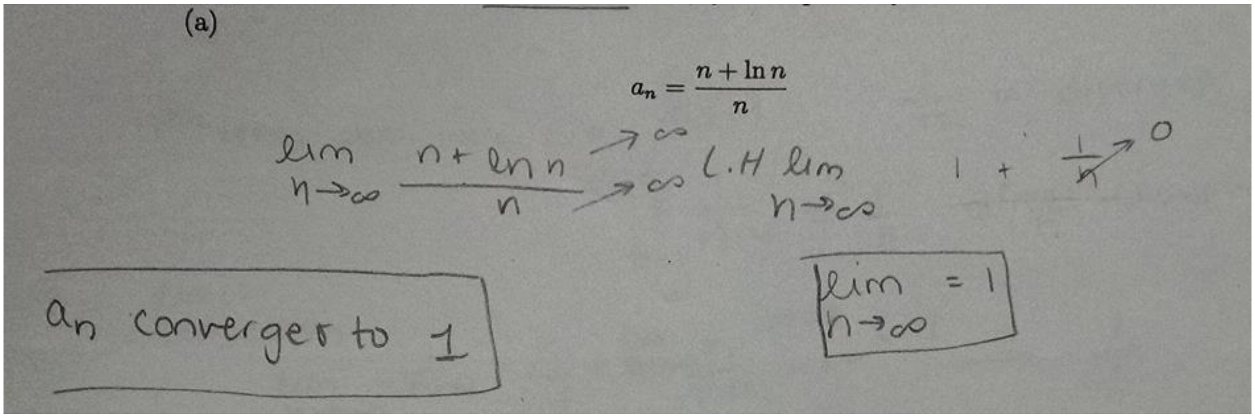
Student Work



In figure 4.2, the student uses an arrow instead of an equals sign in his second line of work. When the limit of a sequence, a_n, is L, we should write either $a_n \to L$ or $\lim_{n\to\infty} a_n = L$. This student combined these notations, indicating a potential misunderstanding of what these symbols mean.

Figure 4.2b: Example of Work with a Notational Error

Student Work

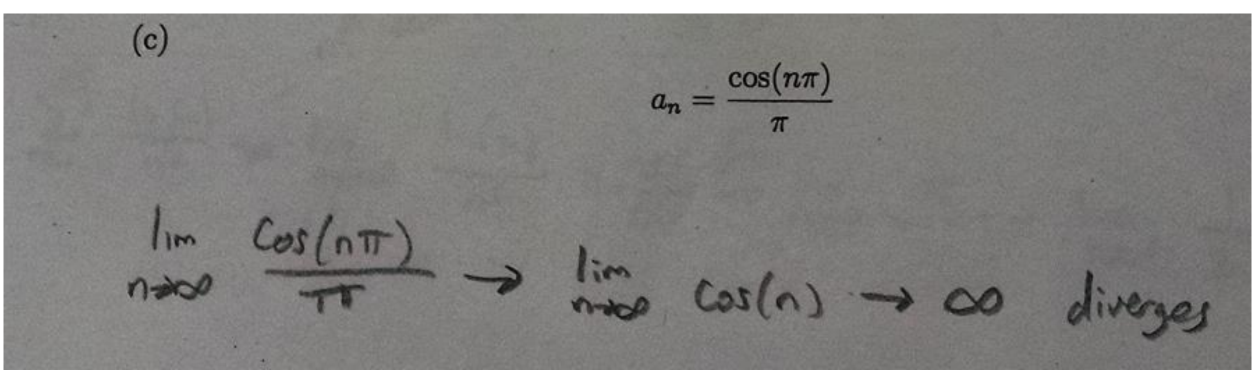


In figure 4.2b, the student does not state the function that he is taking the limit of in his boxed in answer on the right-hand side of his work. One can infer that the student probably meant $\lim_{n\to\infty} a_n = 1$, but the function name is missing. This indicates that the student may have a misunderstanding of the notation used when writing limits.

algebra. Algebra errors included cancelling through a composition of functions, and treating infinity as a real number and plugging it into equations. An example of each can be seen in figures 4.3a and 4.3b:

Figure 4.3a: Example of Work with an Algebra Error

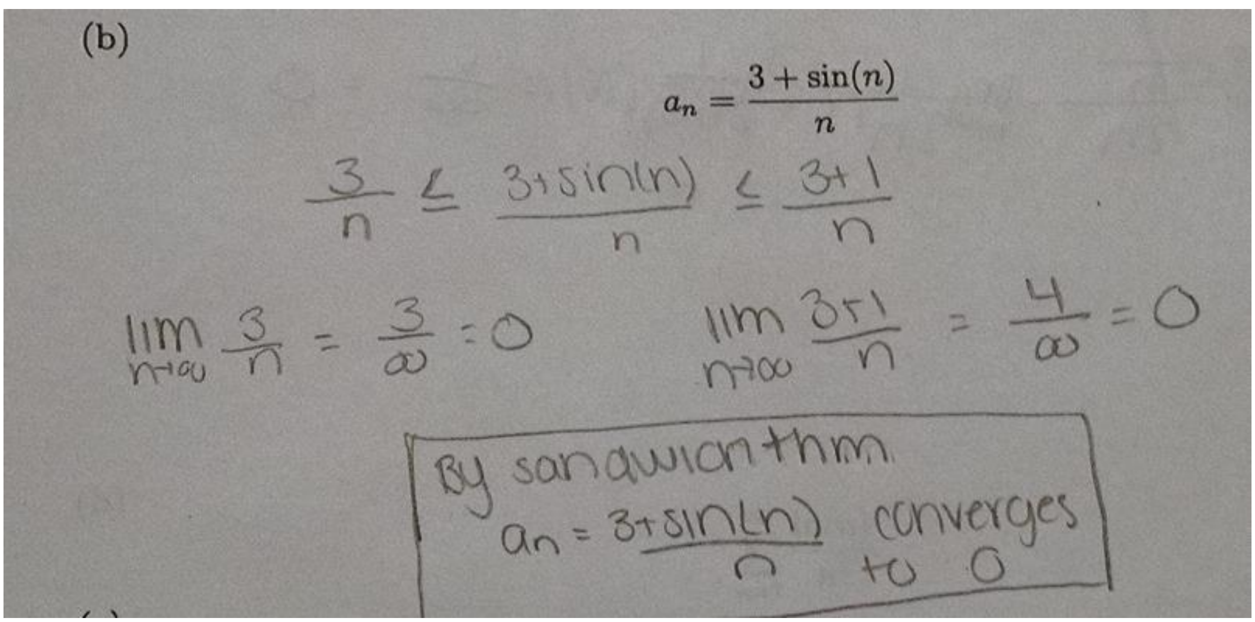
Student Work



In going from the first step to the second step, this student cancels the π in the numerator and the denominator, cancelling through a composition. This indicates that the student may have a misunderstanding about function composition. Note that this student has also made a notational error described earlier, in using an arrow sign in conjunction with the limit notation. This student also does not know the end behavior of the cosine function, stating that it goes to infinity rather than oscillating, indicating a different type of error (misunderstanding of the end behavior of a trigonometric function).

Figure 4.3b: Example of Work with an Algebra Error

Student Work

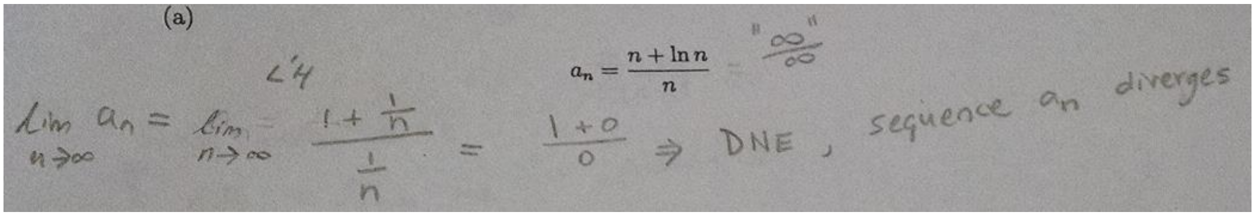


In the second line of this student's work, he plugs in infinity and gets the two equations $\frac{3}{\infty} = 0$ and $\frac{4}{\infty} = 0$. This indicates that the student may think of infinity as a real number.

l'hôpital. Errors categorized by L'Hôpital include the incorrect use of or application of L'Hôpital's rule. Some examples are given in figures 4.4a and 4.4b:

Figure 4.4a: Example of Work with an Error Using L'Hôpital's Rule

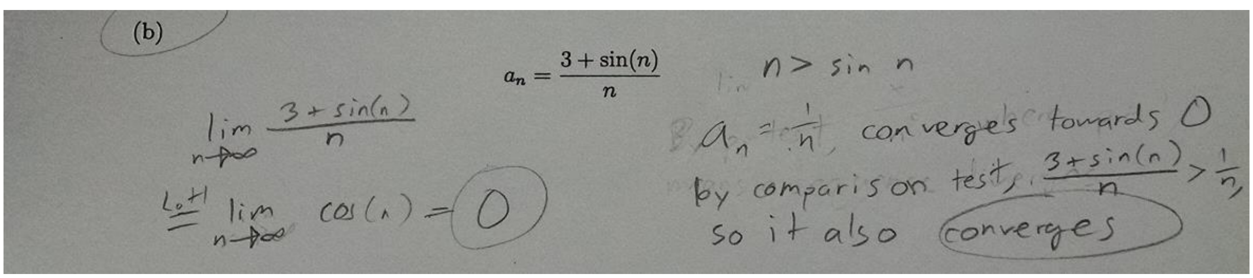
Student Work



The choice to use L'Hôpital's rule here is correct, and the derivative of the numerator is correct. However, the derivative of the denominator should be 1, not $\frac{1}{n}$. It is unclear exactly what the student was thinking here, but perhaps they were mixing up L'Hôpital's rule and the quotient rule. Note that this error leads the student to say that sequence diverges rather than converges to 1.

Figure 4.4b: Example of Work with an Error Choosing L'Hôpital's Rule

Student Work

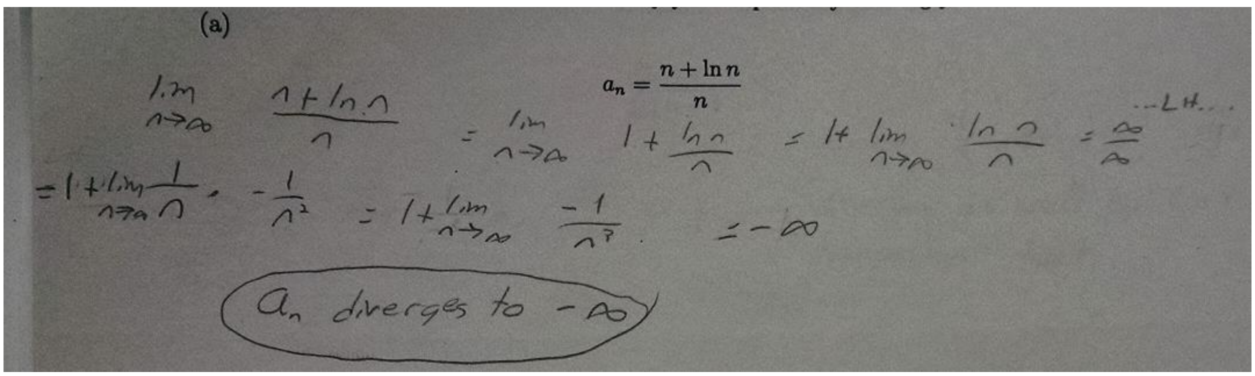


This student chose to use L'Hôpital's rule, but L'Hôpital's rule does not apply here because, while the denominator goes to infinity, the numerator oscillates between 2 and 4. The student might be unsure of the end behavior of the sine function, or he might not know when the rule applies. This student is also incorrect regarding the end behavior of the cosine function, and tries to apply a comparison test which is a series test and cannot be applied to sequences. Thus, in addition to the error in regard to L’Hôpital’s rule, both a trigonometric error and a test invocation error were made.

incorrect behavior argument. Some students made arguments about the numerator in relation to the denominator of a function, or incorrectly indicated the end behavior of a non-trigonometric function. Incorrect end behavior of a trigonometric function was coded under trigonometry. Cases where a student claimed that a sequence diverged to a particular value, such as a claim that a sequence diverged to 0, were also included in this axial category. Examples of student work using an incorrect behavior argument are given in figures 4.5a, 4.5b, and 4.5c:

Figure 4.5a: Example of an Incorrect Behavior Argument

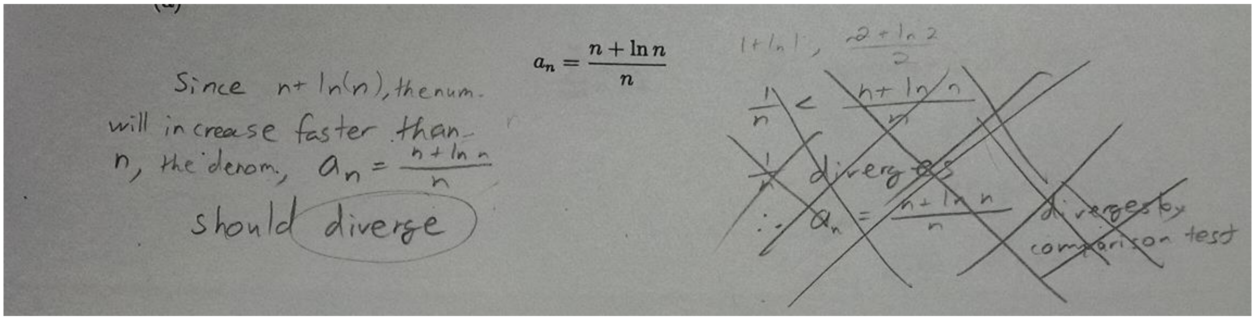
Student Work



This student indicates that the function $\frac{-1}{n^3} \to -\infty$ instead of 0, indicating an incorrect end behavior argument. Note that this student also made an error using L’Hôpital’s rule.

Figure 4.5b: Example of an Incorrect Behavior Argument

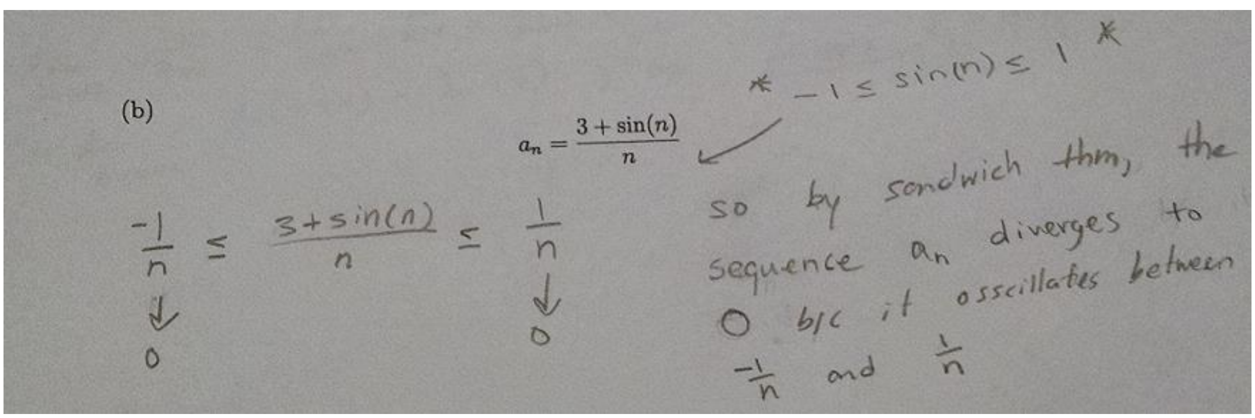
Student Work



Here, the student argues that the numerator grows faster than the denominator, and hence the sequence diverges. Though the numerator is always larger, this argument does not work because as *n* gets large, the sequences approaches 1. In the work that is crossed out, the student tried to use a comparison test, which only applies to series. This type of error, a test invocation error, is discussed in more detail later in this section.

Figure 4.5c: Example of an Incorrect Behavior Argument

Student Work

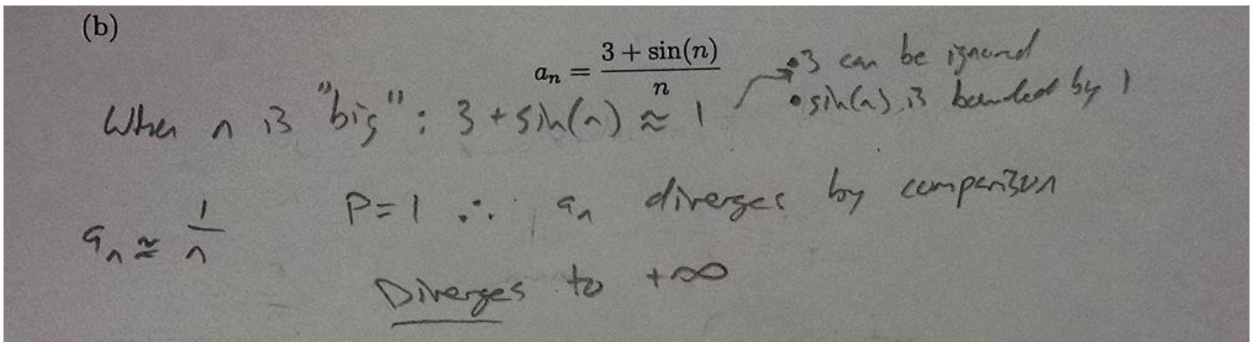


In this third and final example of an incorrect behavior argument, the student concludes that the sequence diverges to 0. This indicates that the student may not understand the difference between the words converge and diverge. Note that this student also made an algebraic mistake, neglecting to add 3 to both sides of the inequality.

test invocation. Test invocation errors were mistakes made by students that involved using a series test on a sequence problem, or a failure to explicitly state the use of the sandwich theorem despite clearly using the theorem. Two examples are given in figures 4.6a and 4.6b:

Figure 4.6a: Example of a Test Invocation Error

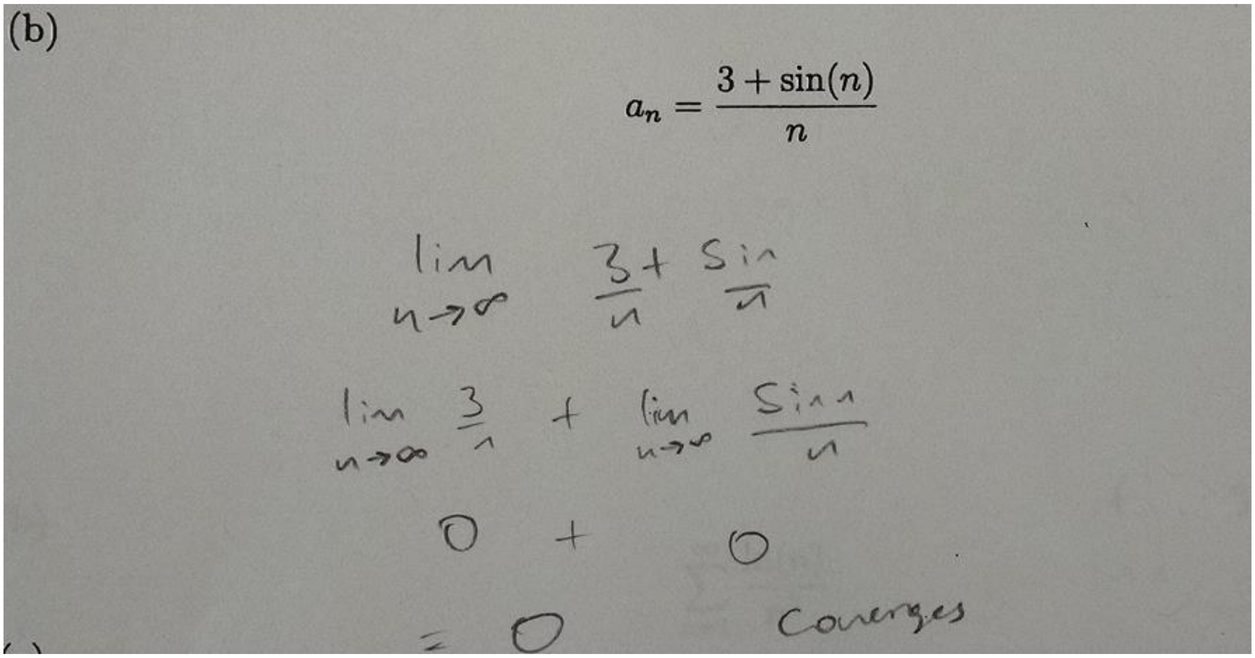
Student Work



This student argues that $3 + \sin n$ is approximately 1, and so the sequence is approximately the same as $\frac{1}{n}$. He then concludes that the sequence diverges by comparison. But comparison tests apply to series, not sequences. This indicates that the student may not know the difference between a comparison test and the sandwich theorem.

Figure 4.6b: Example of a Test Invocation Error

Student Work



This student does not state why $\frac{\sin n}{n} \to 0$. The student should have invoked the sandwich theorem to explain why this statement is true. Consequently, it is unclear if the student knows why $\frac{\sin n}{n} \to 0$.

An example of a test invocation error was also seen during interviews, and can be seen in figure 4.6c, which is from Vinnie's work. Vinnie was working on task 6, trying to find $\lim_{n\to\infty} a_n$.

Figure 4.6c: Example of a Test Invocation Error

Interview Transcript

(1) V: Alright. (long pause) I mean…(pause) Yeah it oscillates, which is why it diverges. So it
(2) con, it grows but it doesn't, it doesn't either reach a solid…ill just put that. Because it slowly
(3) amplifies…or no, I'm drawing it the wrong way, wrong way…I am (inaudible). I'm gonna
(4) say, this goes to zero because…(pause) That's gonna start out as sine, sine of zero equals
(5) that. So it's gonna be… That's what I think the function will do. So…

(6) I: Okay. So you're guessing zero based on the behavior of the graph?

(7) V: Mhmm.

(8) I: Okay.

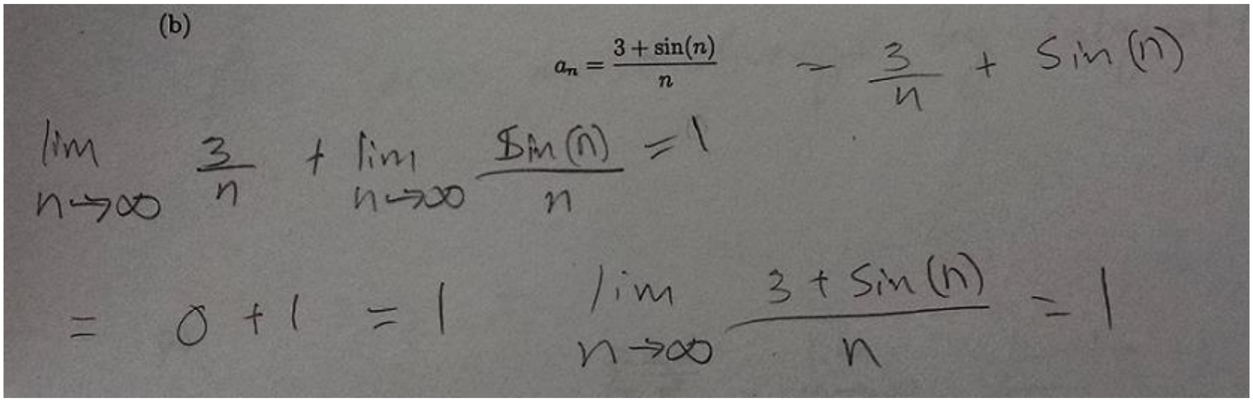
(9) V: It won't ever, it will slowly reach, it will slowly go to zero but…it won't ever reach zero
(10) so…(pause) Yeah I think I'll, I think I'll just leave it like that (laughs)

Vinnie originally thinks the limit will diverge because of the oscillation of the sine function. However, Vinnie is able to graph the function and realize that the graph of the function tends towards zero. Though Vinnie answers the question correctly, he never uses the Sandwich Theorem to make his answer precise. In his work, he even has the necessary inequality (though it is missing the = sign) to use the Sandwich Theorem. Ultimately, he relies on his ability to graph the function to realize that, though the graph is oscillating, the oscillations get smaller and so $\lim_{n\to\infty} a_n = 0$. Vinnie also uses an arrow sign instead of an equals sign, indicating a notational error.

behavior of trigonometric functions. There were a number of errors that students made that fell into this category. One student argued that the cosine function was not continuous. Others believed that both the sine and cosine functions approach infinity rather than oscillating. Still others claimed that $\frac{\sin n}{n} \to 1$. When students believed that the sine and cosine functions go to infinity, they sometimes applied L'Hôpital's rule. Some examples of trigonometry errors can be seen in figures 4.7a and 4.7b:

Figure 4.7a: Example of an Error Involving the Behavior of Trigonometric Functions

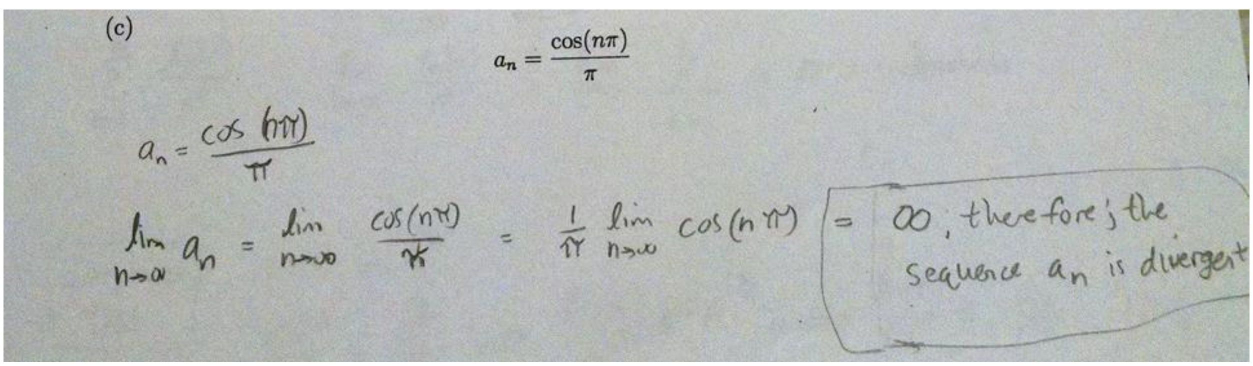
Student Work



Here, the student makes a claim that $\frac{\sin n}{n} \to 1$. However, using the sandwich theorem, one can show that $\frac{\sin n}{n} \to 0$. Hence, it appears the student might not know the sandwich theorem can be used here, or he might be confusing $\lim_{n\to\infty} \frac{\sin n}{n}$ with $\lim_{n\to 0} \frac{\sin n}{n}$.

Figure 4.7b: Example of an Error Involving the Behavior of Trigonometric Functions

Student Work



The student claims that the sequence diverges to infinity because $\frac{1}{\pi}\cos(n\pi) \to \infty$. However, $\cos n\pi$ oscillates between -1 and 1. So while the sequence diverges, it does not diverge to infinity. It appears the student may not understand that the cos function oscillates, and rather thinks that it goes off to infinity.

Students during the interviews also made incorrect behavior arguments about trigonometric functions. Marshall struggles with finding $\lim_{n \to \infty} a_n$ in task 6 below in figure 4.7c.

Figure 4.7c: Example of an Error Involving the Behavior of Trigonometric Functions

Interview Transcript

(1) M: Okay. The limit as n goes to infinity of…this. This, since it is…uh, a trig function, it's
(2) going to oscillate between some two numbers. It's not gonna be negative one and one cuz
(3) you have all these other numbers in there. (clears throat) But, it's going to oscillate.
(4) So…we'll see about the limit.

(5) I: So, because it's a trig function, any trig function would have the same property.

(6) M: Umm…specifically sine and cosine.

(7) I: Okay.

(8) M: Because tangent isn't going to oscillate. It just going to go from negative infinity to
(9) positive infinity. Or, pi over two, sorry. So…(long pause) I guess the limit doesn't exist?

(10) I: And why is that?

(11) M: Umm…because it'll just continuously oscillate to infinity, between whatever two
(12) numbers it's bounded by in the y direction.

Marshall is correct in identifying that sine and cosine functions will oscillate. However, he claims that it will not oscillate between -1 and 1 because, "you have all these other numbers in there." Marshall also appears to disregard the *n* in the denominator, indicating that he thinks the oscillation of the sine function guarantees divergence. Marshall showed no written work for this problem, only writing an answer of, "the limit of the sequence D.N.E. because it will oscillate between its bounds in the y-direction infinitely."

Series. Most of the errors students made for series were captured by the axial categories identified in the pilot study problem 4b described in the methodology section. As a result of interviews, a new category developed, Confusing Sequences and Series (CSS). The other axial categories were, Notational Error (NE), Algebra of Series (AS), Algebra (A), Function Choice (FC), Unchecked Assumptions (UA), Algebra error leading to Incorrect Test Choice (AITC), Incorrect Test Choice (ITC), Wrong Conclusion Drawn from Test (WCDT). Table 4.8 summarizes these categories. In the next few subsections, these categories are described in more detail with more examples from student work.

Table 4.8: Axial Categories, Abbreviations, and Explanation Table for Series
Axial Categories, Abbreviations, and a Brief Explanation with an Example for Series

Category	Abbreviation	Explanation
Confusing Sequences and Series	CSS	Students that confused sequences and series often used the words interchangeably, and in some cases were inconsistent using summation notation. Also includes students that seemed unsure of the definition of sequences and series
Algebra	A	An algebraic error. A student might, for example, "plug in" infinity, or incorrectly simplify a rational expression by "cancelling" through a sum
Function Choice	FC	Wrong function choice when using a comparison test. For example, a student might try to make a comparison with $\frac{1}{n^2}$.
Unchecked Assumptions	UA	Student failed to check that the function satisfied the assumptions in the integral test.
Algebra error leading to Incorrect Test Choice	AITC	Student reaches a false conclusion (usually in the ratio test) because of an algebraic mistake. This mistake typically was cancelling through a sum.
Incorrect Test Choice	ITC	Student chooses an incorrect test, such as an nth term test, or a geometric test.
Wrong Conclusion Drawn from Test	WCDT	Student reaches an incorrect conclusion from using a series convergence test. For example, a student uses the ratio test and says that a value of 1 means the series converges.

confusing sequences and series. During the interviews, and on the first question in particular, students would use the words sequence and series incorrectly, using the word sequence when they meant series and vice versa. An example is given from Moira's attempt at solving task one in figure 4.8a. Line numbers have been added for reference later.

Figure 4.8a: Example of Confusing Sequences and Series

Interview Transcript

(1) Interviewer: Okay. You can go ahead. And this first question I did not label very well. But its
(2) two separate questions. The first one is about this series, a sub n, and the second one is
(3) involving b sub n.

(4) Moira: Okay. (pause) Okay. So…am I, do I write it? Am I just writing it, or do you want me
(5) to tell you?

(6) I: Tell me and write it, yeah.

(7) M: Okay. Well…since the limit of this is a number, we know that…the sequence is going to
(8) be…divergent because this approaches a number other than zero.

(9) I: Can you say that again? The sequence is gonna be divergent because…

(10) M: Suppose I know…is a-n a series, or, is it just a…I was thinking a-n, since the limit of the
(11) series approaches a number by the nth term test, since it doesn't equal zero, you know that
(12) the sequence diverges.

(13) I: Okay.

(14) M: Assuming a-n is a series. (laughs) Shoot. I know that in this one, on this one, when I did
(15) it before, I like thought something and then I switched it at the last second. Suppose you
(16) know that (inaudible). Hm. Okay. Yeah so, by nth term test. Then this one…since it equals a
(17) number, it converges. So the limit of this would be zero.

(18) I: Just for clarification, what is 'this?'

(19) M: Um, the limit of the series?

(20) I: Okay.

(21) M: Because then by nth term test, that would converge. Well you don't know it actually
(22) converges, but it has to be zero if this is gonna converge.

(23) I: Okay.

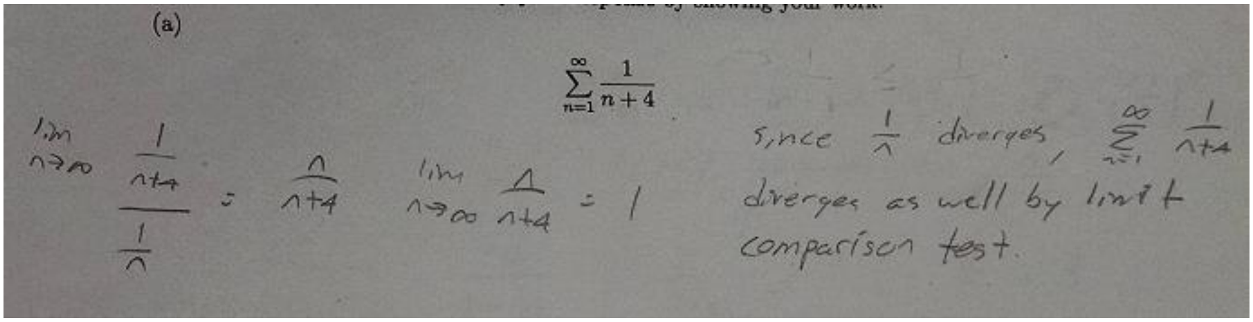
It is first worth noting that Moira's written answer was correct. However, during our discussion, she seemed confused at several points about what was a sequence and what was a series. The first bit of confusion shows up in line 7, where Moira talks about the sequence being divergent because "this" (here she refers to the $\lim_{n\to\infty} a_n$) approaches a number other than three. However, she means (and writes) that the series $\sum_{n=1}^{\infty} a_n$ diverges. In line 10, she is unsure if a_n is a sequence or a series, ultimately incorrectly deciding it is a series. In lines 11 and 12 she again mixes up the use of the words sequences and series (where she said sequence she should have said series and vice versa). In line 19 she again uses the word series when she is referring to the limit of the sequence, b_n. Moira would try to avoid using the terminology altogether by pointing to what was already written as "this" and "that." However, when pressed by the interviewer, it was clear she had confused the terminology.

Moira is an interesting case because, though she has mixed up the meaning of the words, she is otherwise able to reason through the problem correctly. She refers to the nth term test when needed, and ultimately writes down the correct solution. Moira shows that even students that have a solid understanding of the concepts can be confused by the terminology.

notational error. Student work that used incorrect notation fell into this category. For example, a student did not include a series sign when referring to a series. Two examples are given in figures 4.9a and 4.9b:

Figure 4.9a: Series Notational Error

Student Work



(a)

$$\sum_{n=1}^{\infty}\frac{1}{n+4}$$

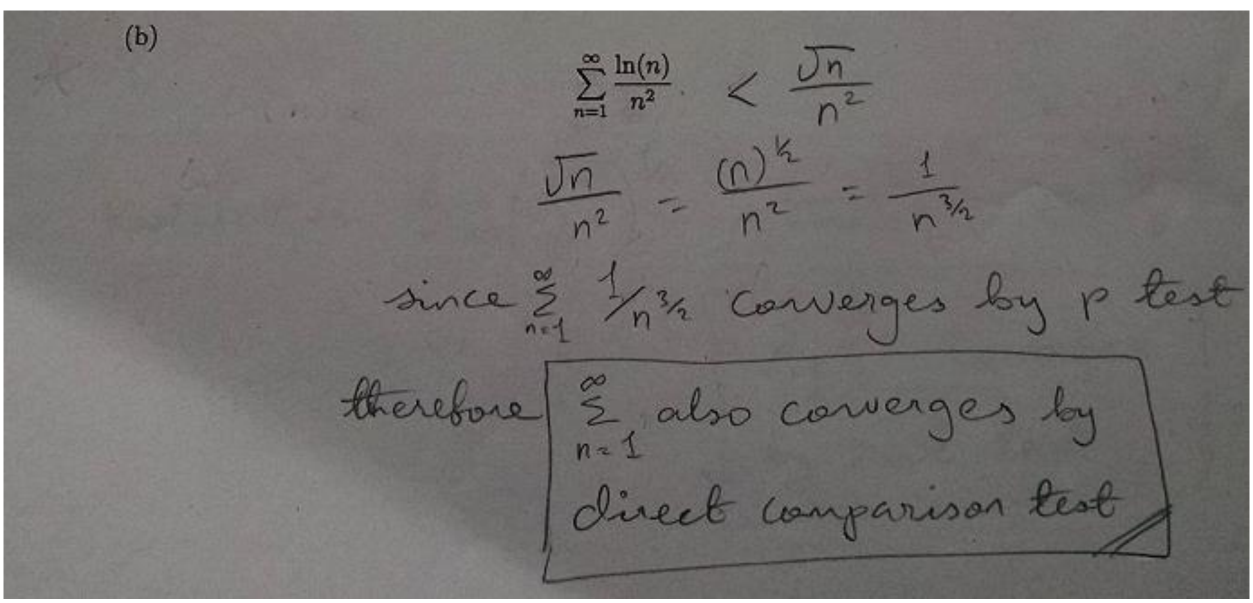
$$\lim_{n\to\infty}\frac{\frac{1}{n+4}}{\frac{1}{n}} = \frac{n}{n+4} \qquad \lim_{n\to\infty}\frac{n}{n+4} = 1$$

since $\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}\frac{1}{n+4}$ diverges as well by limit comparison test.

While this student took a correct approach to solving the problem, there are a couple of notational errors. First, the limit sign is missing in the second step. It should read $\lim_{n\to\infty}\frac{n}{n+4}$. Second, the series sign is missing in the statement $\frac{1}{n}$ diverges. It should read $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. It is unclear if the student is not sure which notation to use, or if perhaps the student was rushed for time and just forgot to include the proper signs.

Figure 4.9b: Series Notational Error

Student Work



(b)

$$\sum_{n=1}^{\infty}\frac{\ln(n)}{n^2} < \frac{\sqrt{n}}{n^2}$$

$$\frac{\sqrt{n}}{n^2} = \frac{(n)^{1/2}}{n^2} = \frac{1}{n^{3/2}}$$

since $\sum_{n=1}^{\infty} 1/n^{3/2}$ converges by p test

therefore $\boxed{\sum_{n=1}^{\infty} \text{ also converges by direct comparison test}}$

This student also has made a couple of notational errors. First, in the very first step, the student compares the series to a sequence, rather than just comparing the sequences. This could be indicative of a lack of understanding about the difference between a sequence and a series. The first step should read $\frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2}$ for *n* sufficiently large. Finally, at the end of the problem, the student writes a series sign without indicating the object he is taking the infinite series of. He should have written $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ also converges…

There were also instances where incorrect notation was used during interviews. For example, using arrows where an equals sign is more appropriate, or not writing the limit sign. But one student, Peter, offered some insight on why he used the notation he did, which might help explain some of the other notational errors that were seen. The excerpt in figure 4.9c is from task 5.

Figure 4.9c: Series Notational Error

Interview Transcript

(1) I: Okay. So I'm just gonna ask one quick question.

(2) P: Of course

(3) I: Uh on here you started with a series, and then switched to a limit sign.

(4) P: Um okay so I just wanted to take, I wanted to just take the limit

(5) I: Okay

(6) P: Of this sequence.

(7) I: Okay

(8) P: Throughout each step. Um I try to be good about writing limit signs, like I do on the test.
(9) My own personal, when I do my homework and stuff, I don't

(10) I: Okay

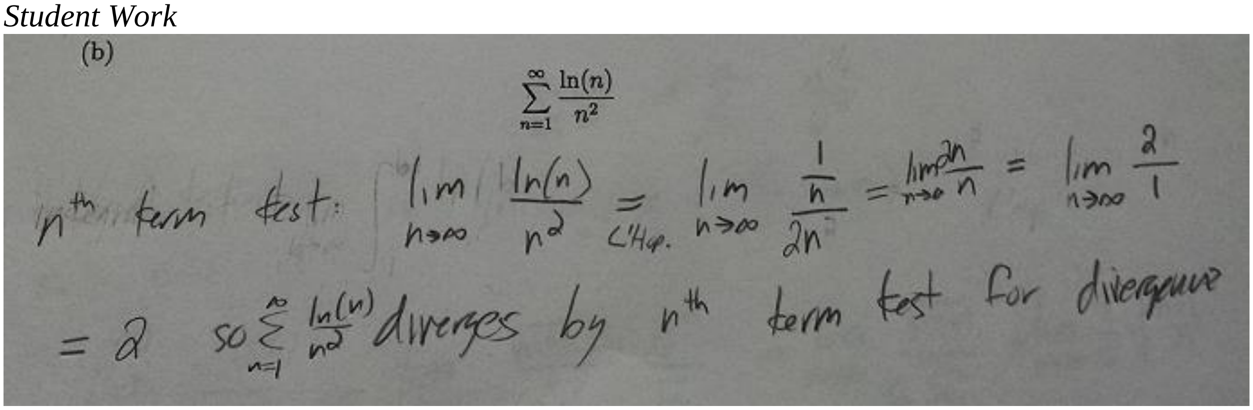
(11) P: I don't write the limit sign out each time because frankly I just think it's a pain in the ass.

In his work, Peter starts with a series sign, and then switches to a limit sign, using arrows between each step. Peter seems to think that notation is not needed, calling it a "pain in the ass" that he does not use in his own personal work. From his work, it can also be seen that Peter does not have a solid understanding of the p-test, and even starts the series at n=0 rather than n=1.

algebra. Algebra errors included mistakes simplifying exponents, rational expressions, and the distributive property. Some algebra errors had little impact on a student's ability to solve a problem. For example, failing to distribute a negative sign generally did not alter a student's answer on whether or not a series converged or diverged. However, mistakes involving the simplification of rational expressions and exponents could have a large impact on whether or not a student thought a particular series converged or diverged. Two examples are shown in figures 4.10a and 4.10b:

Figure 4.10a: Example of an Algebra Error

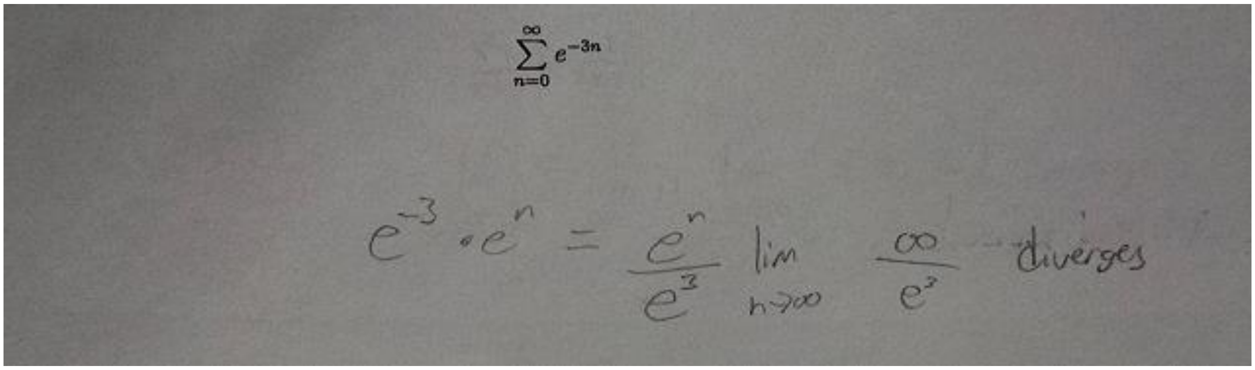
Student Work



Here, the student tries to use the nth term test to solve the problem. However, the student incorrectly simplifies $\frac{\frac{1}{n}}{2n}$ to $\frac{2n}{n}$ instead of $\frac{1}{2n^2}$. Thus, the end result for the student is that the series diverges by the nth term test, rather than stating the nth term test is inconclusive and moving on to another test. The student's inability to simplify a rational expression prevented him from solving the problem correctly.

Figure 4.10b: Example of an Algebra Error

Student Work

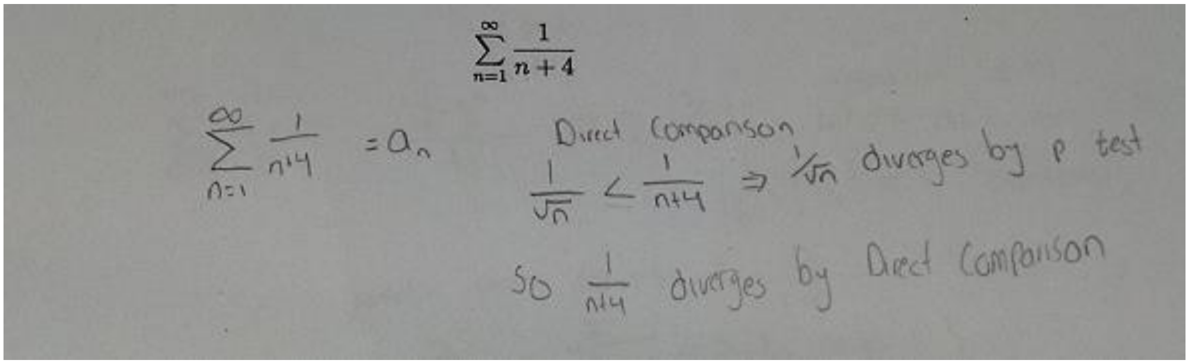


This student appears to simplify e^{-3n} as $\frac{e^n}{e^3}$, which is incorrect. The student was perhaps thinking of the expression e^{n-3}, or else may not understand how to simplify exponents. This simplification leads the student to conclude that the series diverges since $e^n \to \infty$. This series is actually a convergent geometric series. Note that there is also a notational error here, as the student does not indicate the object he is taking the limit of.

function choice. When using a comparison test, students would sometimes choose a function that would not help them solve the problem, or chose a function that was difficult to use. Two examples of this are given in figures 4.11a and 4.11b:

Figure 4.11a: Example of an Incorrect Function Choice

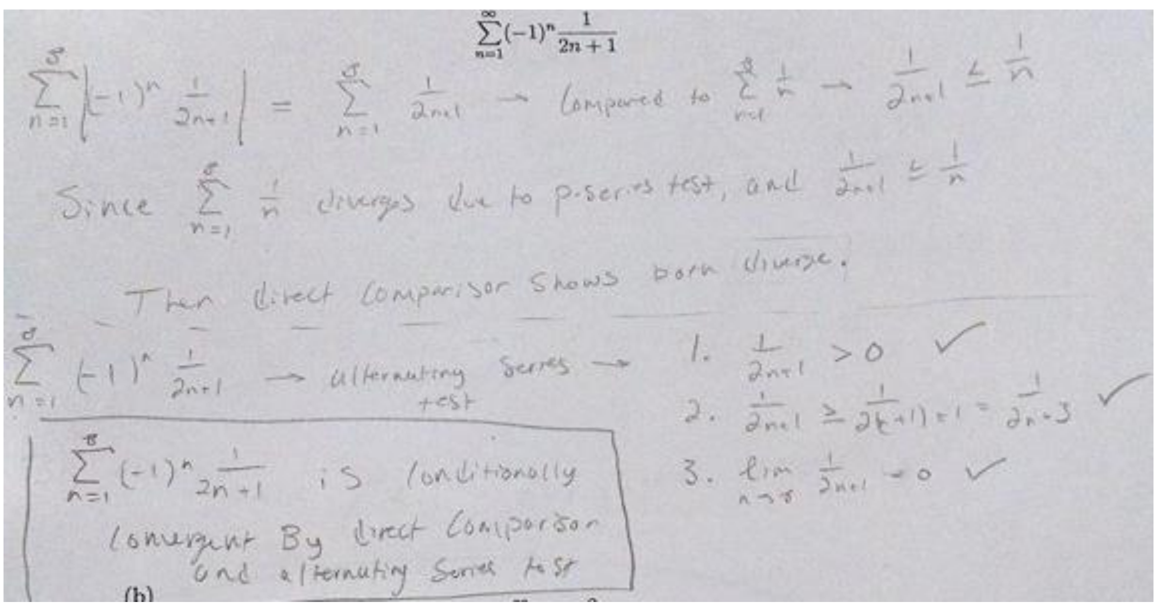
Student Work



This student makes an incorrect comparison and uses this comparison to conclude that the series diverges. It is unclear why the student chose the function $\frac{1}{\sqrt{n}}$ instead of $\frac{1}{n}$. Perhaps he thought that the comparison was accurate because $\sqrt{n} < n + 4$. Maybe he thought he needed a smaller exponent in the denominator to make the comparison.

Figure 4.11b: Example of an Incorrect Function Choice

Student Work

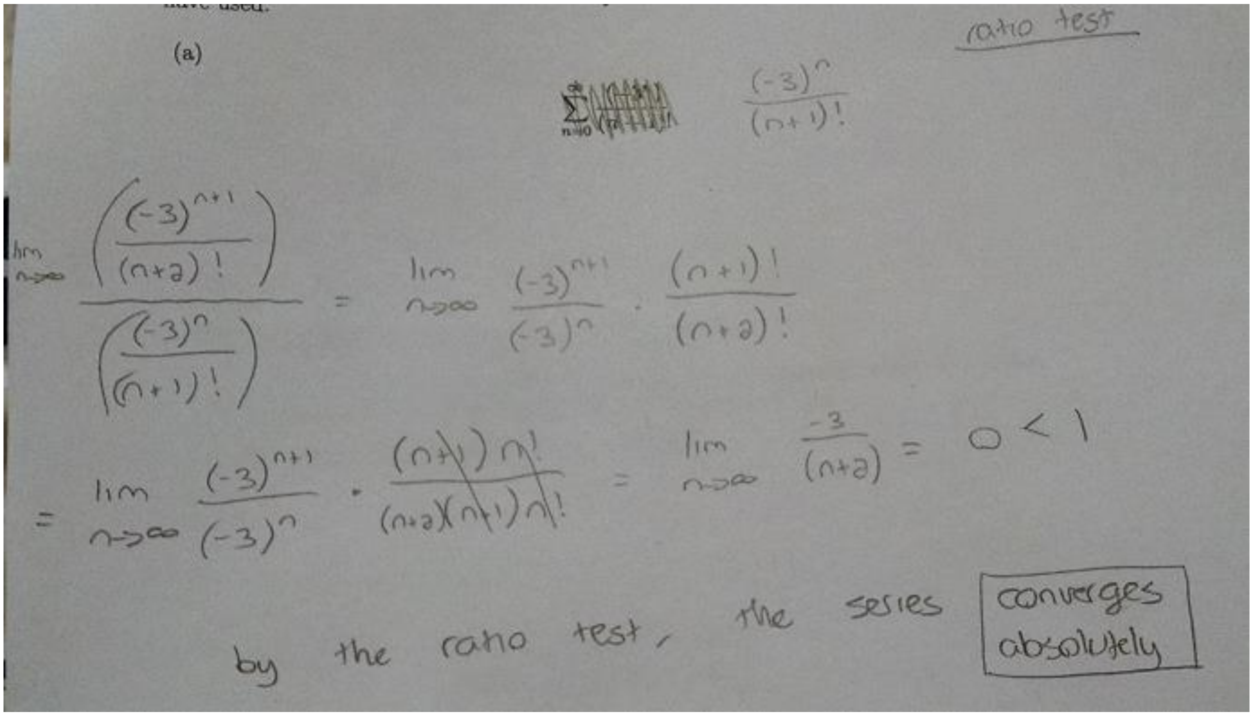


While testing for absolute convergence, this student notices that $\frac{1}{2n+1} \leq \frac{1}{n}$. This comparison is correct, but knowing this comparison and that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges doesn't result in the divergence of $\sum_{n=1}^{\infty} \frac{1}{2n+1}$. It is unclear why the student thought the comparison $\frac{1}{2n+1} \leq \frac{1}{n}$ and the fact that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges would lead to any conclusion using the direct comparison test about the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2n+1}$.

unchecked assumptions. Some students neglected to check that the assumptions for using a test were satisfied. For example, some students didn't check for positivity when using a comparison, root, or ratio test. Other students didn't make sure that the function was non-increasing and continuous when using the integral test. Two examples are given in figures 4.12a and 4.12b.

Figure 4.12a: Example of Unchecked Assumptions

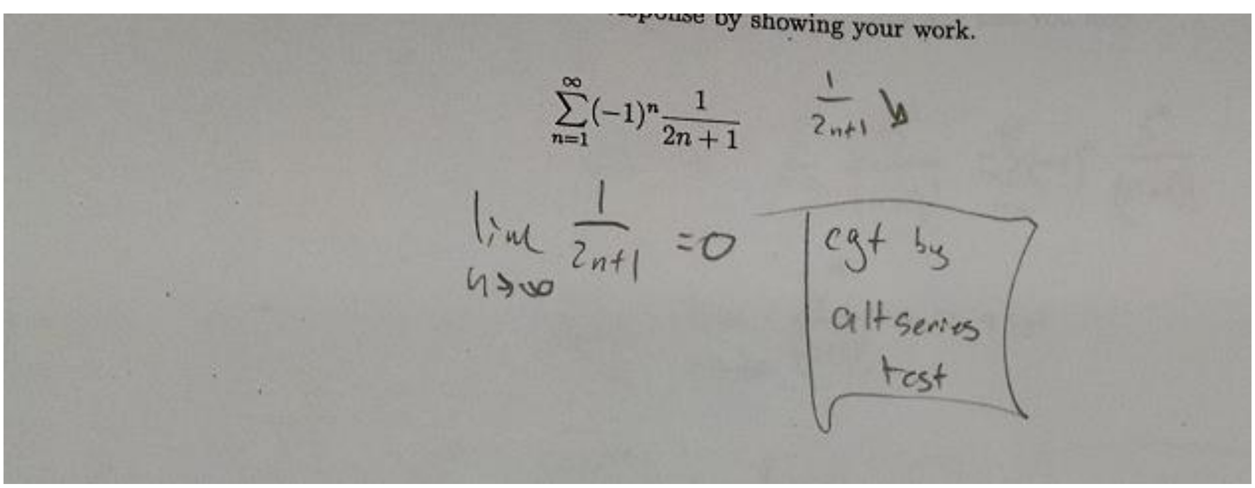
Student Work



This student's work indicates that he wants to use the ratio test to solve this problem, but the ratio test can only be used on series with positive terms. Thus, this student failed to check that the conditions of the ratio test were satisfied.

Figure 4.12b: Example of Unchecked Assumptions

Student Work



This student checked one of the assumptions in the alternating series test, but failed to check the other two assumptions. In particular, the student's work does not indicate that he checked that $\frac{1}{2n+1}$ is positive and nonincreasing. The student also doesn't explain why this series is alternating, but it is perhaps obvious with the $(-1)^n$ term in the series.

Students also failed to check the assumptions of the direct comparison test during interviews. An example from Moira's solution to task 6 is given in figure 4.12c:

Figure 4.12c: Example of Unchecked Assumptions

Interview Transcript

(1) M: Okay. So we know that by the p-test this, okay we're gonna use this side, this, we know
(2) that by the p-test this is gonna diverge. So, you want to use the left side. Because since this
(3) diverges and this is greater than it it's gonna make that diverge as well. And if you use the
(4) right side, it wouldn't tell you anything because this would be diverging, but it doesn't, this
(5) could be any number. (long pause)

(6) I: And is this a particular test you're using? Or is this just…

(7) M: I was just about to write that.

(8) I: Okay. Sorry.

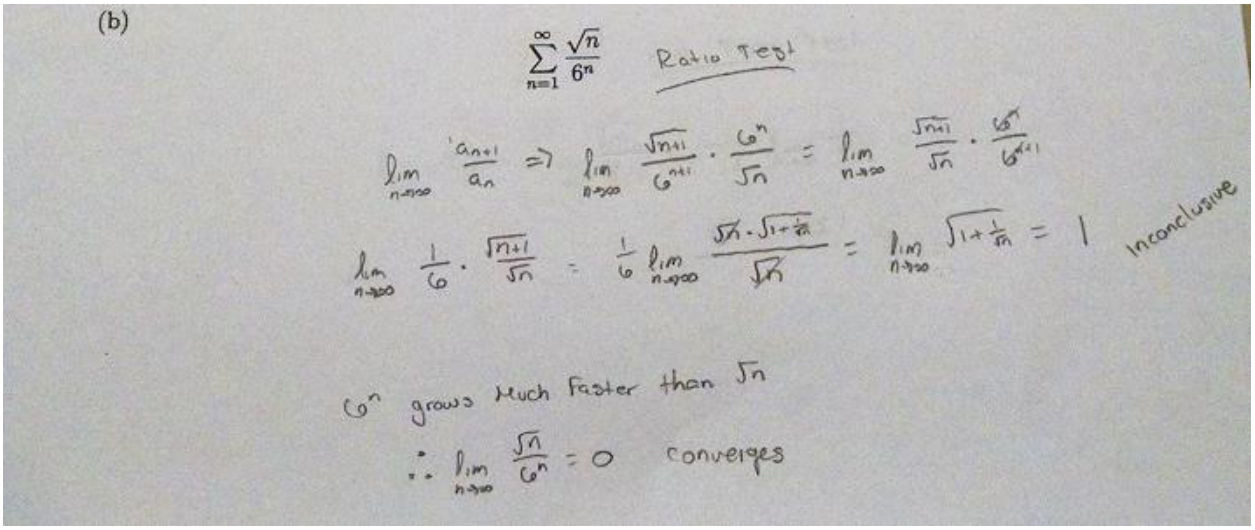
(9) M: Umm, no it's cool. Diverges…by direct comparison test.

The "left side" in line 2 refers to $\frac{-1}{n}$, and she notices that $a_n > \frac{-1}{n}$, and concludes that the series diverges by the direct comparison test. Moira's reasoning about the direct comparison is correct, but her comparison is invalid because she fails to check the assumptions in the direct comparison test. In particular, the series that she is comparing must both be positive.

algebra error leading to incorrect test choice. Sometimes, algebra mistakes led students to reach an incorrect conclusion regarding which test to use. An example is given in figure 4.13:

Figure 4.13: Example of an Algebra Error Leading to an Incorrect Test Choice

Student Work

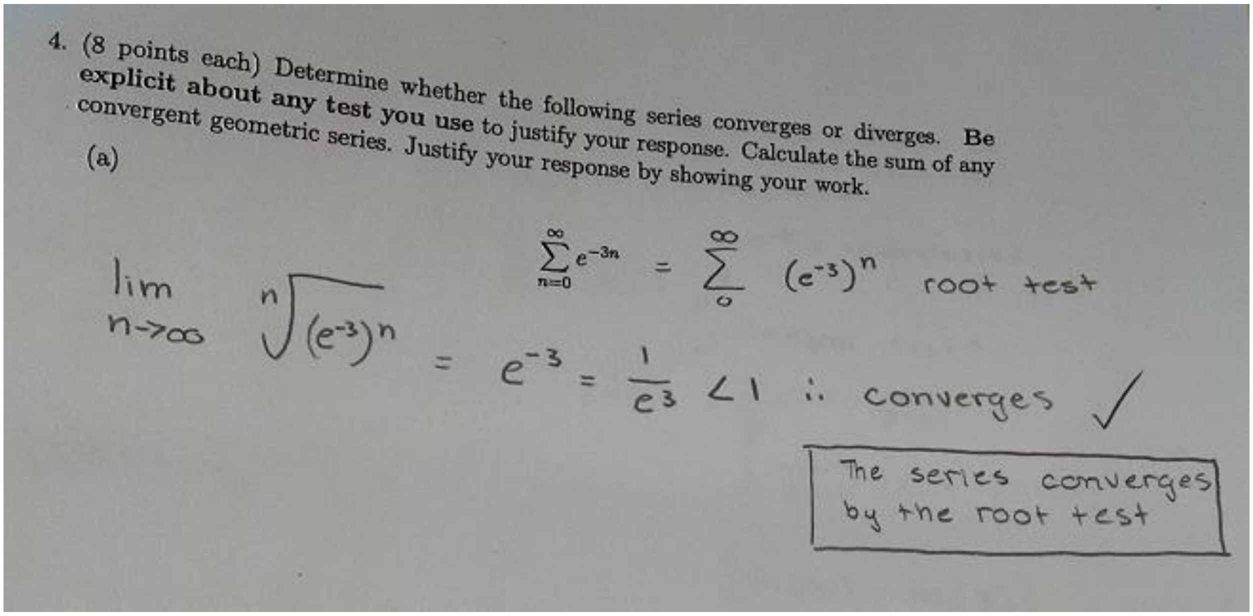


Here, the student starts off trying to use the ratio test. Using the ratio test, the student should have gotten a limit value of $\frac{1}{6}$. However, once the student pulled the $\frac{1}{6}$ outside of the limit sign, he neglected to multiply his final answer by $\frac{1}{6}$. Consequently, the student got a value of 1 in the ratio test and found the test inconclusive. The student then uses the nth term test, which should be inconclusive, and determines that the series converges.

incorrect test choice. Sometimes, students used a convergence test that could not help them solve the given problem. An example is given in figure 4.14a:

Figure 4.14a: Example of an Incorrect Test Choice

Student Work



4. (8 points each) Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work.

(a)

$$\sum_{n=0}^{\infty} e^{-3n} = \sum_{0}^{\infty} (e^{-3})^n \quad \text{root test}$$

$$\lim_{n\to\infty} \sqrt[n]{(e^{-3})^n} = e^{-3} = \frac{1}{e^3} < 1 \quad \therefore \text{ converges}$$

$\boxed{\text{The series converges by the root test}}$

The student's use of the root test in this problem is correct. However, the instructions clearly state to find the sum of any convergent geometric series. The root test cannot be used to find the sum of the series. In other words, the work indicates that the student may have failed to recognize this series as a geometric series.

Another example of an incorrect test choice comes from the interviews. One student, Edith, tried to use partial fractions on task 3, thinking the series was telescoping. She was asked why she thought it was telescoping, and her answer is seen in figure 4.14b:

Figure 4.14a: Example of an Incorrect Test Choice

Interview Transcript

(1) I: So let me go back a little ways here. What made you originally think this was a telescoping
(2) series by the way? You mentioned it early on and I probably should have asked you then why
(3) you thought that, but…

(4) E: Um because not everything in it was being raised to a power. Like in the last one, the
(5) geometric series everything in it was being raised to a power. Whereas this one, you can see
(6) that you need to use partial fractions which all the examples that I practiced for the test we

(7) had to, the telescoping series ones, we had to use partial or, is it called partial or impartial
(8) fractions?

(9) I: Partial fractions.

(10) E: Partial fractions, partial fractions.

(11) I: So the only two types of series you have are geometric and telescoping?

(12) E: For ones that you are finding the sum of. And this wants you to find, if it converges,
(13) explain how you would go about finding the sum. So that's why I said…since it's asking for
(14) a sum that it would need to be telescoping or geometric.

(15) I: Okay, what if the problem did not say that? What would you have done then? What if the
(16) problem just said 'determine if the series converges or diverges?' What would you have
(17) done?

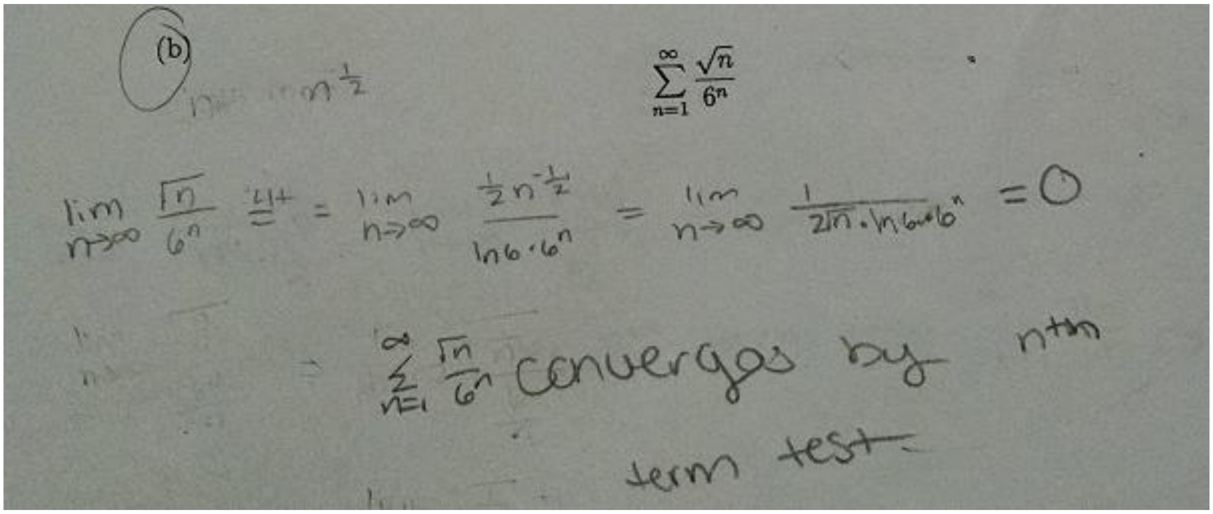
(18) E: Hmm. Nth term test wouldn't have worked because it would have gone to zero because
(19) you have n over n squared since this, when you multiply out, the highest one would be n
(20) squared. So that wouldn't have worked. So I probably would have done a comparison test.

The wording of the problem appears to be effecting Edith's thinking. In lines 12-13, she thinks that since the series converges, and she is being asked to find the sum, it must be telescoping or geometric. However, the question does say, "or explain why you cannot find the sum." Once prompted to answer the problem with different directions in line 16, Edith proceeded to use the limit comparison test to correctly determine that the series diverges.

wrong conclusion in test. Student solutions that fell into this category involved choosing a series test to solve a problem and then reaching an incorrect conclusion using that test. Four examples are given in figures 4.15a-4.15d:

Figure 4.15a: Example of a Wrong Conclusion in Test

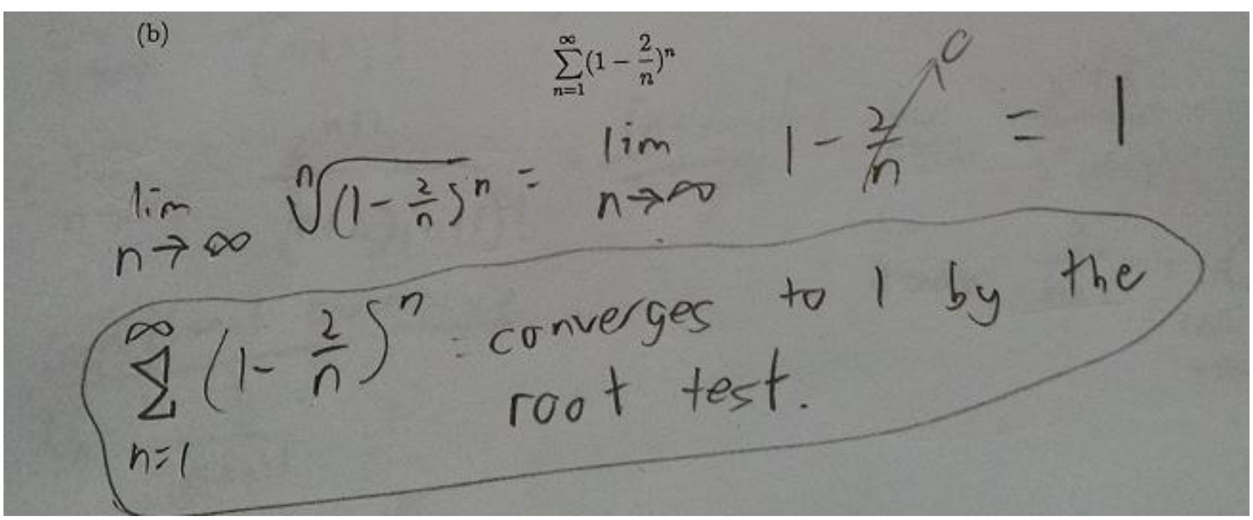
Student Work



This student tries to use the nth term test to solve this problem when either a ratio test or root test should be used. When the student discovers that the limit of the sequence is 0, he concludes that the series converges. It appears the student thought that a value of 0 in the nth term test meant that the series converges.

Figure 4.15b: Example of a Wrong Conclusion in Test

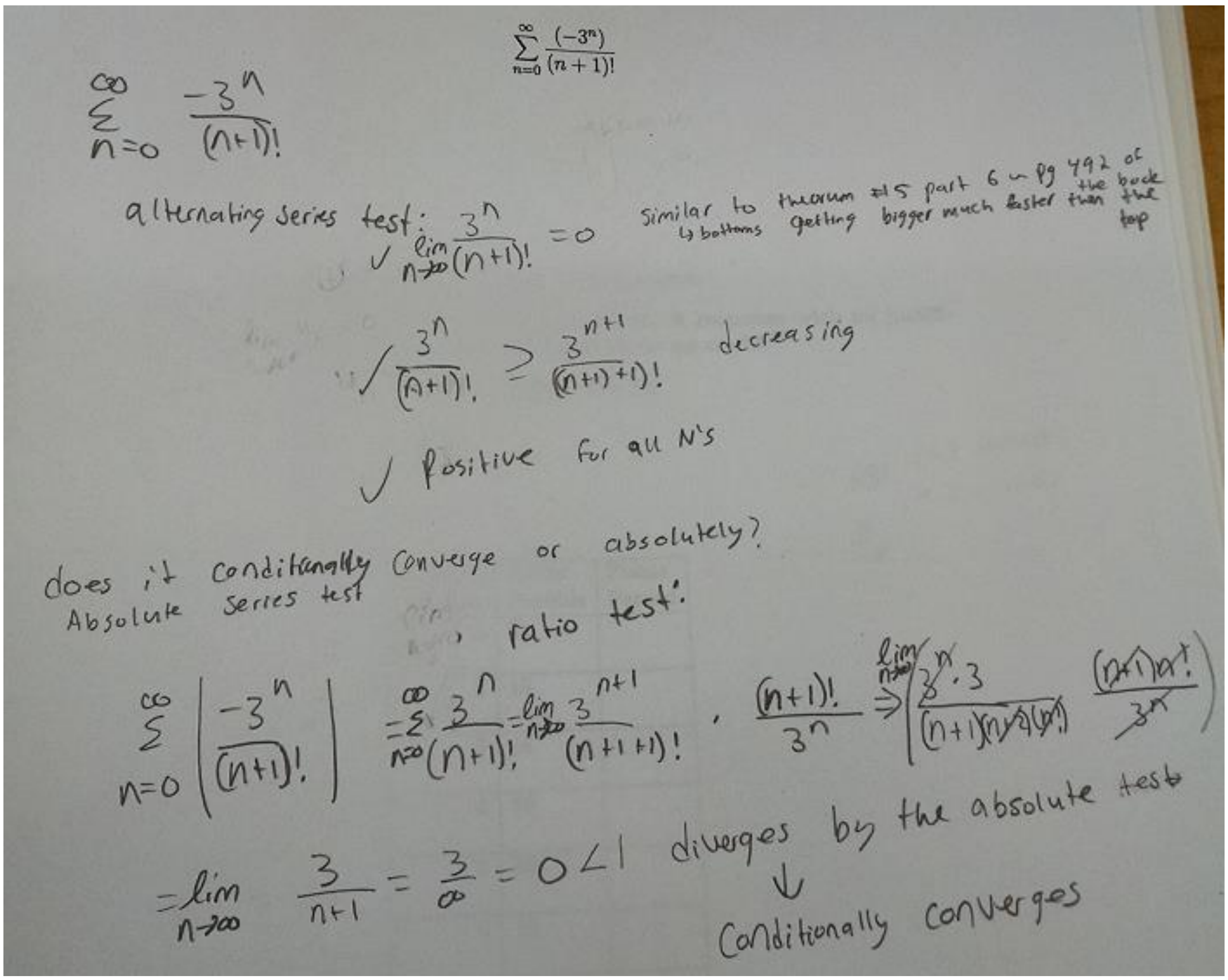
Student Work



The student uses the root test. He performs the test correctly and gets a value of 1, which should indicate that the test is inconclusive. However, the student instead concludes that the series converges to 1, indicating that he may think the value obtained at the end of the test is the value of the infinite series.

Figure 4.15c: Example of a Wrong Conclusion in Test

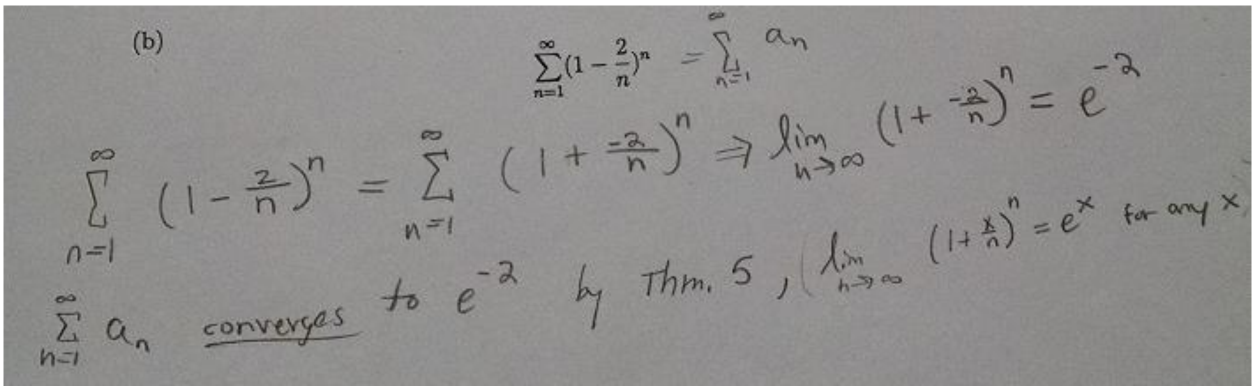
Student Work



When testing for absolute convergence, this student correctly uses the ratio test, and then uses the test correctly, coming up with a limit of 0. However, this should tell the student that the series converges absolutely. The student instead indicates that the series diverges by the absolute test, and then goes on to test for conditional convergence, indicating that he may not know what to conclude when using the ratio test.

Figure 4.15d: Example of a Wrong Conclusion in Test

Student Work



This student appears to take the correct approach to solving the problem, taking the limit of the sequence and discovering that it is e^{-2}. However, instead of concluding that the series diverges by the nth term test, the student instead indicates that the series converges to this value.

Students also reached incorrect conclusions from series tests in interviews as well. An example is given in figure 4.15e from Marshall's solution to task 2.

Figure 4.15e: Example of a Wrong Conclusion in Test

Interview Transcript

(1) I: Okay. And can you say anything about what the sum could be?

(2) M: The sum would be zero.

(3) I: The sum of the series is zero?

(4) M: Yes.

(5) I: Okay. And why is that? Because that's what you got from the ratio test? Or is there another
(6) reason why the sum is zero?

(7) M: Hmm. (long pause)

(8) I: I guess what I'm asking is if we had taken this limit and we had gotten a value of, say, one-
(9) half here, would that mean that the series converges to one-half?

(10) M: Uhh…yes. I'm gonna go ahead and say yeah.

(11) I: Okay.

(12) M: So yeah. I guess the sum is zero because it's what the ratio test tells me.

(13) I: Okay.

The ratio test is the correct approach for solving task 2, but the value of 0 in the ratio test tells us that the series converges, but not what the series converges to. Marshall seems confused by this, and is unable to reason through why the sum being 0 would be absurd. For example, there are many series with all positive terms where the limit of the ratio would be 0. But Marshall seems convinced that, "the sum is zero because it's what the ratio test tells me."

Qualitative Results Summary

The difficulties students had on the sequences portion of the exam included notational errors, algebra mistakes, incorrect use of L'Hôpital's rule, incorrect arguments about the behavior of functions, failure to invoke the name of a test (or use the correct test), and mistakes involving trigonometric functions. Student difficulties with series included confusing the words, notational errors, unjustified adding of series that don't converge, algebra errors, difficulties choosing a function to use in the direct comparison test, failure to check that the assumptions of a test are not satisfied, choosing a test that won't help them solve the problem, and an incorrect interpretation in what the conclusion of a series says.

The way these results address the research questions is discussed in chapter five.

Chapter 4 – Quantitative Analysis and Results

The purpose of this chapter is to describe the results of the analysis of the multiple-choice assessment. A brief summary of the number of participants is given, followed by the results of the exploratory and confirmatory data analysis.

Phase 3 – Multiple Choice Items

A total of 185 students responded to the multiple-choice test. However, six responses were incomplete, and so only 179 responses were used in data analysis. Recall that the analysis for this phase consisted of first exploratory data analysis (EDA) followed by confirmatory data analysis (CDA). This section first describes the results from EDA, followed by a description of the results from CDA. All fractional percentages were rounded to the nearest whole number.

Exploratory Data Analysis (EDA). Students in this study were between 18 and 33 years of age. The majority of students (82%) were either 18 or 19 years old. Most students were freshman (76%), male (75%), and white (86%). Some of the students surveyed had taken AP calculus in high school (29%), while a few took precalculus in high school (15%). Under half of the sample (38%) claimed to have seen sequences and series before in a previous course. Very few students (7%) did not know what their grade in the course would be at the time of the survey or thought they would not pass the course (D or F).

Table 5.1 summarizes how students performed on each question on the assessment. The table lists the question number along with the percentages that answered each question correctly and incorrectly.

Table 5.1 – Overall Percentages for Questions 1-6

Question Number and Percentage Response Correct and Incorrect

Question Number	Percentage Correct	Percentage Incorrect
1	32%	68%
2	54%	46%
3	67%	33%
4	77%	23%
5	58%	42%
6	27%	73%

Students had the most success on problem four, and the least success on problems one and six. On problem one, many students (40%) chose incorrect answer choice D. On question six, over half (51%) chose incorrect answer choice C.

Recall that option weighting was used in the overall scoring of the exam, with 1 point awarded for correct answers, a half a point awarded for partially correct answers, and no points awarded for totally incorrect answers. The average score on the exam was just over a 4 (M = 4.14, SD = 0.95). The minimum score was a 1.5, and the maximum score was a 6.

Looking at two variables at a time, such as experience with sequences and series and total score, the only consideration was whether or not a student answered the question correctly, and not individual answer choices. This is because, when performing CDA, comparing individual answer choices resulted in contingency tables with too few values, and JMP warned of inaccuracies in statistical tests.

A number of results stood out when comparing two variables. First, students that had experience with sequences and series in the past were more likely to answer most of the questions correctly. Students without experience outperformed students with experience only on question four. These results are summarized in Table 5.2.

Table 5.2 – Overall Percentages for Questions 1-6 Based on Experience With Sequences and Series

Question Number and Percentage Response Correct and Incorrect

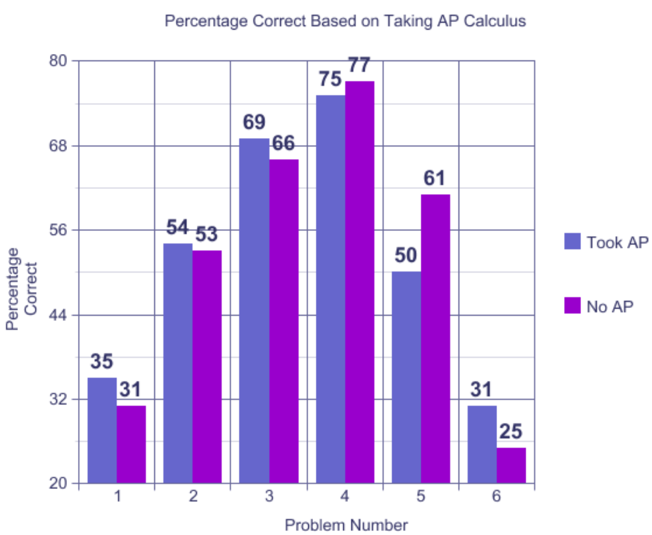
Question Number	**Percentage Correct With Experience**	**Percentage Correct Without Experience**
1	38%	29%
2	63%	48%
3	69%	66%
4	72%	79%
5	62%	55%
6	37%	21%

In terms of overall score, students without experience had an average slightly over 4 (M = 4.05, SD = 0.89). There was also a score of 1.5 which was a statistical outlier. 1.5 was the minimum score and 6 was the maximum. Students with experience had a slightly higher average

(M = 4.3, SD = 1.04). There were no statistical outliers. The minimum score was 2 and the maximum was 6.

Students that took AP calculus (either AB, BC, or both) slightly outperformed students that had not taken AP calculus on questions one, two, three, and six. Those that had not taken any form of AP calculus outperformed students that had on questions four and five. These results are summarized in Figure 5.3. In terms of overall scores, students that did not take AP calculus had an average slightly over 4 (M = 4.14, SD = 0.98). Students that took some form of AP calculus had a similar average (M = 4.15, SD = 0.89).

Figure 5.3 – Percentage Correct on Each Problem Based on Having Taken AP Calculus



Students that took precalculus at this research institution were outperformed by students that passed out of the precalculus course via a placement test on every question. This information is summarized in Table 5.4.

Table 5.4 – Overall Percentages for Questions 1-6 Based on Taking Precalculus

Question Number and Percentage Response Correct and Incorrect

Question Number	Percentage Correct: No Precalculus at this Institution	Percentage Correct: Took Precalculus at this Institution
1	33%	29%
2	56%	43%
3	68%	61%
4	78%	68%
5	60%	46%
6	28%	18%

In terms of overall scores, students that did not take precalculus at this institution had and average over 4 (M = 4.21, SD = 0.92), while students that did take precalculus at this institution had an average less than 4 (M = 3.79, SD = 1.05.

Students that answered question six correctly were more likely to get the other five questions correct than students that did not answer question six correctly. A total of 48 students answered question six correctly, while 131 did not. The students that answered question six correctly outperformed students that did not answer question six correctly on question one (52% to 25%), question two (60% to 51%), question three (81% to 62%), question four (77% to 76%), and question five (71% to 53%).

Before looking across two variables when one of the two variables was expected grade, some options had to be lumped together, or else there were again warnings from JMP about the accuracy of statistical tests. For example, only 2 students said they expected to get an A/B in the course, as opposed to an A or a B. In such borderline cases, the researcher decided to consider the higher of the two, and so any student that said A/B was put into the A category, and students that said B/C were put into the B category. The third category was C or lower, and the four students who said they don't know and the one student who left this part of the survey blank were not included in any category. Consequently, of the 174 students that gave a valid response, about a quarter (23%) said they expected an A in the course, roughly half (49%) said they expected a B in the course, and a little over a quarter (28%) said they expected a C or lower in the course. The results based on expected grade in the course are summarized in Table 5.5.

Table 5.5 – Expected Grade and Percentage Correct

The Percentage of Students that Answered Correctly Based on Expected Grade

Expected Grade	**Question 1 Correct**	**Question 2 Correct**	**Question 3 Correct**	**Question 4 Correct**	**Question 5 Correct**	**Question 6 Correct**
A	50%	75%	78%	88%	78%	35%
B	29%	53%	62%	74%	51%	27%
C or less	24%	39%	65%	69%	53%	20%

It is worth noting that students that expected an A outperformed students that expected a B and a C or less on every problem. The students that expected a B outperformed students that expected a C or less on four of the six problems.

In terms of overall score, students that expected an A had a high average (*M* = 4.71, *SD* = 0.84). Students that expected a B had a lower average than those that expected an A, just over 4

(M = 4.05, SD = 0.93), and students that expected a C or lower had an average under 4 (M = 3.81, SD = 0.90).

As a result of EDA, several hypotheses developed. These hypotheses were:

1. Students that had experience with sequences and series prior to entering the course performed better than those that did not have experience. Note that this experience varied. Some students who responded that they had experience noted that their experience came from seeing finite sequences and series in precalculus. Others recognized infinite series from calculus I and seeing the definition of the integral.
2. Students that took precalculus at this institution performed much lower than students that placed out of precalculus.
3. Students that answered question six correctly were more likely to answer the other five questions correctly. CDA will confirm for which problems the difference was statistically significant.
4. Finally, it appeared as though students that were confident in their performance in the course (expecting an A) performed much better than those that were not as confident (expecting a B, or a C or lower), as seen in Table 5.4.

All of these hypotheses were tested during CDA.

Confirmatory Data Analysis (CDA). The purpose of CDA is to use statistical tests to determine if the hypotheses made after EDA are statistically significant. As mentioned in Earls (in preparation), Pearson chi-squared tests were used for nominal variables based on recommendations from the Institute for Digital Research and Education website (IDRE, n.d.), and two-sample t-tests were used when one variable was nominal and one was continuous. As was true during EDA, statistical tests were performed using the JMP software.

Hypothesis One. For students that said they had experience with sequences and series, their performance on questions two, χ^2 $(1, N = 179) = 4.067, p = .044$, and six, χ^2 $(1, N = 179) = 5.531, p = .019$ were significantly better than those that did not have experience. Recall that students without experience performed slightly better than students with experience on question four. However, this difference was not statistically significant, χ^2 $(1, N = 179) = 1.224, p = .269$. There was also no statistical significance to performance based on experience in questions one, three, and five. Finally, overall scores for students with experience were not significantly better than scores for students without experience, $t(177) = 1.75, p = .081$.

Hypothesis Two. Though students that took precalculus at this university were seen during EDA to perform worse than students that did not take precalculus at this university, the differences in performance on each of the six questions was not statistically significant. The differences in total score were also not statistically significant, $t(177) = -1.99, p = .054$, though the power was only 0.5810. Hence, it is possible that there is too much noise in the data hiding statistical significance. One possible reason for this is that there were only 28 students surveyed that took precalculus at this university. Hence, the sample size might be too small to determine statistical significance.

Hypothesis Three. Students that answered question six correctly were significantly more likely to answer questions one, χ^2 $(1, N = 179) = 11.599, p < .001$, three, χ^2 $(1, N = 179) = 5.994, p = .014$, and five, χ^2 $(1, N = 179) = 4.743, p = .029$, correctly.

However, there was no such correlation between answering questions two, $\chi^2\ (1, N = 179) = 1.214, p = .271$, and four, $\chi^2\ (1, N = 179) = 0.011, p = .917$, correctly.

Hypothesis Four. Expected grade in course appeared to show a strong correlation with answering questions one, $\chi^2\ (2, N = 174) = 7.351, p = .025$, two, $\chi^2\ (2, N = 174) = 11.713, p = .003$, and five, $\chi^2\ (2, N = 174) = 8.602, p = .014$ correctly. However, there did not appear to be a significant correlation between expected grade in course and answering questions three, four, and six correctly at the 0.05 level. Finally, in terms of overall score and expected grade in the course, students that expected an A in the course performed significantly better than both the students that expected a B in the course, $t(171) = -4.70, p < .001$ and the students that expected a C $t(171) = -3.83, p < .001$. However, the difference in total score between students that expected a B and those that expected a C or lower was not statistically significant at the 0.05 level, $t(171) = -1.48, p = .140$.

Phase 3 Summary

Students performed better on question four than any other question on the assessment, and they struggled the most with question six. Prior experience with sequences and series played a role in student success on the assessment, as did the expectation of getting an ‘A’ in the course. Students that did manage to answer question six correctly had more success on three other questions than students that did not answer question six correctly. Hypothesis one was not significant in turns of overall scores, but was significant for certain items. There were not enough respondents to determine the significance of hypothesis two. Hypothesis three was true for some items, but not for others. Hypothesis four was significant. This is discussed in more detail in the next chapter.

Chapter 5 – Discussion, Implications, and Conclusions

This chapter relates the results of the study back to the three research questions stated in the introduction, explains how the results contribute to the existing literature summarized in the literature review, describes changes that need to be made to the optimal concept map, frames the results in terms of instrumental and relational understanding described in the theoretical perspective in chapter one, and discusses the implications for further research on student understanding of sequences and series.

Results and Research Questions

For the reader's convenience, the 3 research questions are stated again here:

1. What misconceptions of sequences and series are revealed when students solve problems on sequences and series typically seen in a second semester calculus course?
2. In what ways, if at all, do these misconceptions relate to the prerequisite knowledge students are expected to have prior to starting a second semester calculus course?
3. What additional understanding or conceptualization of sequences and series might students need to be successful in second semester calculus courses?

The next sections discuss how the results address each of the three research questions.

Research Question One. The methodology section described the difference between errors and misconceptions in this study. In particular, persistent errors were deemed to be misconceptions.

Table 6.1 shows the type of error students made and whether there is evidence that the error is persistent according to the criteria in the previous paragraph.

Table 6.1: Area in Which Student Made an Error, and Whether the Error is Persistent

Type of student error, and is it persistent

Type of Student Error	Evidence the Error is Persistent?
Determining End Behavior of Sequences	No
Notation	Yes
L'Hôpital's Rule	Yes
Using and Choosing Appropriate Tests	Yes
Trigonometric Sequences	Yes
Algebraic Simplification of Exponential and Rational Functions	Yes
Assumptions for Series Tests	Yes
Conclusions in Series Tests	Yes
Differences Between Sequences and Series	No
Contrapositive of the Nth Term Test	Yes
Recognizing Geometric Series	Yes

The reasoning behind classifying errors as persistent is given in the following subsections.

determining end behavior of sequences. The errors involving determining the end behavior of sequences were only seen in the exam data. Consequently, there is insufficient evidence to claim this error is persistent.

notational errors. Notational errors were seen in the exam data and during interviews. For example, in the exam data, students wrote $\lim_{n\to\infty} a_n \to L$, and $\lim_{n\to\infty} = 1$. During interviews, a student wrote $\sum a_n \to \lim_{n\to\infty} a_n$. Since exam data was collected in Spring 2015 and full interviews took place in Spring 2016, notational errors are viewed as persistent, and hence there is evidence to suggest that students have misconceptions about the proper notation to use when dealing with sequences and series. More specifically, students seemed to have misconceptions about when and how to use a limit sign and summation notation.

l'Hôpital's rule. The errors that students made involving L'Hôpital's Rule only occurred in this study on exam data. For example, students took the incorrect derivative of the function $f(x) = x$, saying that $f'(x) = \frac{1}{x}$. Errors involving taking derivatives were seen in first semester calculus courses in Orton's (1983) study. Consequently, errors involving taking derivatives are persistent errors across first and second semester calculus, providing some evidence that students have misconceptions about derivatives of linear functions. Students also made errors involving the use of L'Hôpital's Rule. For example, they applied it to trigonometric functions that were not of the correct form. These types of errors are discussed in the section on trigonometric sequences.

using and choosing appropriate tests. Errors that involved using and choosing appropriate tests were seen in both interviews and exam data. During interviews, Edith was convinced she was dealing with a telescoping series because the problem asked her to explain how to find the sum. She seemed to miss the part of the question that said, if she could not find the sum to explain why. Thus, it appears an incorrect reading of the question led Edith to believe she needed to show the series was telescoping. In the exam data, students used a root test on a geometric series when asked to find the sum of the series. Consequently, these errors are classified as persistent. In other words, students appear to have misconceptions about how to choose and use an appropriate test for a given problem. More specifically, students seem to rely on the tests they find easiest to work with, rather than tests that are best suited to solving the given problem.

trigonometric sequences. Errors about the end behavior of trigonometric sequences were seen in the exam data and during interviews. In addition, Weber (2005) found that students in precalculus had difficulty determining where the sine function is decreasing, indicating that they may have difficulties understanding the oscillation of the sine function. Consequently, errors about the end behavior of trigonometric sequences appear to be persistent. Misconceptions about the end behavior of trigonometric sequences from this study include thinking that the sine and cosine functions tend towards infinity, and that $\frac{\sin n}{n} \to 1$.

algebraic simplification of exponential and rational functions. Cangelosi et al. (2013) found that students had persistent errors across college algebra, precalculus, calculus I, and calculus II when it came to simplifying exponential expressions with a negative exponent. Exam data and pilot study interviews revealed that students thought of e^{-3x} as e^{x-3} when trying to determine if $\sum_{n=1}^{\infty} e^{-3x}$ converges, and what the series converges to. Consequently, evidence from this study and from Cangelosi et al. (2012) suggest that students have misconceptions about the simplification of exponential functions with a negative exponent.

Though errors simplifying rational functions were only seen in exam data, Makonye and Khanyile (2015) found that students in high school algebra also made similar mistakes simplifying rational functions; in fact, several of the students in their study cancelled through a sum which was also seen in the exam data in this study. Thus, there is evidence to suggest that students have misconceptions about simplifying rational functions.

assumptions for series tests. Students failed to check that the assumptions for certain series tests were satisfied in both the exam data and during interviews. In the exam data, students failed to show that they checked the positivity of the ratio test and the three conditions in the alternating series test. During interviews, a student failed to check the positivity condition for the direct comparison test. Hence, evidence suggests that students have misconceptions about checking assumptions when using series tests. More specifically, it appears students do not understand the importance of the assumptions in series tests.

conclusions in series tests. Students made errors in determining what they should conclude from a series test on both the exam data and during interviews. In many cases, they thought that the value obtained when using a series test told them what the series converged to. Because the errors occurred in both the exam data and during interviews, the errors are classified as persistent.

differences between sequences and series. Student confusion about the difference between a sequence and a series was only seen during interviews. Consequently, the error cannot be classified as persistent, and evidence does not support that there is a misconception here. However, it is worth noting that the nature of this error required asking students for clarification while they were working. Thus, it is possible that students did not know the difference between sequences and series on the exam, but it was impossible to know by only looking at their work. Further research is needed to determine whether these errors could be classified as misconceptions.

contrapositive of the nth term test. In this study, errors involving the contrapositive of the nth term test were only seen in the multiple-choice assessment. More precisely, 73% of the students surveyed were unable to identify the contrapositive of the nth term test when answering question six. Gregg (1997) found that students and teachers in geometry have difficulty understanding the logical equivalence between a statement and its contrapositive. This means that there is evidence to suggest that students have misconceptions about the contrapositive of a statement.

recognizing geometric series. Students made errors recognizing a series as geometric in both the exam data and on the multiple-choice assessment. On the exam, students failed to recognize $\sum e^{-3n}$as a geometric series. On the multiple-choice assessment, 47% of those surveyed failed to recognize $\sum \pi^n\, e^{-2n}$ as a geometric series. Thus, there is evidence to suggest that students have misconceptions about identifying geometric series. More precisely, students may not recognize a series as a geometric series when the ratio involves irrational numbers.

Research Question Two. Recall that research question two asks how the misconceptions found in question one relate to prerequisite knowledge. Table 6.2 lists the persistent errors classified as misconceptions found in this study and how these misconceptions might be related to prerequisite knowledge. Note that depending on the specific courses taken, it is possible students have not been exposed to certain topics. For example, though students might learn about the equivalency of a statement and its contrapositive from high school geometry, it is possible that students might not have seen this topic before. However, the purpose of this table is to show where students might gain this beneficial prerequisite knowledge.

Table 6.2: Misconception and Prerequisite Knowledge

Name of misconception and what prerequisite knowledge is needed

Misconception	Beneficial Prerequisite Knowledge
Notation	Limit Notation from Calculus I
L’Hôpital’s Rule	Derivatives of linear functions from Calculus I
Using and Choosing Appropriate Tests	Computing Limits from Calculus I, simplifying functions
Trigonometric Sequences	End behavior of trigonometric functions (especially sine and cosine) from precalculus

Algebraic Simplification of Rational and Exponential Functions	Knowledge of rational and exponential functions from and precalculus
Assumptions for Series Tests	Checking hypotheses of theorems are satisfied (for example, checking differentiability in the Mean Value Theorem in Calculus I)
Conclusions in Series Tests	Verifying conclusions of theorems (for example, confirming that a ‘c’ value exists in the Mean Value Theorem, even though it does not tell us how to find the ‘c’)
Contrapositive of the Nth Term Test	Knowledge of the equivalence of a statement and its contrapositive from geometry
Recognizing Geometric Series	Knowledge of properties of exponents from precalculus to identify ratios

A few things about the data in this table are worth mentioning. First, having the prerequisite skills in the table does not guarantee that the misconceptions listed will be avoided. For example, students should have experience checking that the assumptions of the Mean Value Theorem are satisfied in first semester calculus. This does not guarantee that a student that has success checking assumptions in first semester calculus will have no problems checking assumptions for series tests in second semester calculus. The assumptions for series tests are different, and more knowledge about sequences and series may be needed to be successful. What more a student may need to know is discussed in the section on research question three.

Second, the beneficial prerequisite knowledge was compiled based on the errors that were seen in this study. For example, the misconceptions about L’Hôpital’s Rule in this study centered around the problems students had taking the derivatives of linear functions. But there are other types of functions that students need to differentiate to be successful using L’Hôpital’s Rule. However, students in this study did not show difficulties differentiating these other types of functions, and consequently, there may be more beneficial prerequisite knowledge than is listed in the table. For example, students need to be able to correctly take the derivative of the natural log function.

Finally, the name of the course where students are expected to have acquired the beneficial prerequisite knowledge is based on the course sequence at the University where this study took place. The course sequence leading up to calculus II is precalculus followed by calculus I. Since courses vary from school to school, students may learn about these topics in different courses than those that are listed in the table. For example, the results suggest that students should have a knowledge of the properties of exponential and rational functions from precalculus. However, at some colleges and universities, these properties may be covered in college algebra, a course that is not offered at the University where this study took place.

It is also worth mentioning that there are many prerequisite skills that students should have from courses taken even prior to high school. For example, to be able to simplify rational functions in precalculus, it is important for students to be able to understand how to add and multiply fractions. Understanding integer exponents as repeated multiplication can help students when they are trying to simplify exponential functions. In other words, there are prerequisite skills necessary to master the topics in precalculus and calculus mentioned in this section.

Research Question Three. The third research question asks what else students need to know to be successful in second semester calculus. This section discusses how relational understanding might help students be more successful in second semester calculus courses.

Skemp (1976/2006) defined relational understanding as, “knowing both what to do and why” (p. 89). He contrasts this with instrumental understanding, which he describes as, “rules without reasons” (p. 89). Skemp argues in favor of relational understanding over instrumental understanding because relational understanding is adaptable to new problems and is easier to learn since mathematics is seen as a connected whole.

There is evidence from the current study to suggest that students have an instrumental understanding, but not a relational understanding. As an example, consider one misconception students have about using and choosing appropriate tests. Students seemed to have memorized a rule that, if a factorial appeared in the series, then the ratio test was the best to use. Evidence that students have memorized this rule can be seen in the multiple-choice assessment number three, which most (67%) of the population answered correctly. In addition, students had little to no trouble with interview task four, which asked students to determine the convergence of a series containing a factorial. However, in general, students had difficulty knowing which test to use in which situations. For example, during the pilot study interviews, some students stated that they started with the ratio test on a problem involving a rational function simply because it was the easiest for them to simplify. Another student in the full study interview was choosing which test to use because she thought the question demanded finding the sum. Consequently, it appears students are learning instrumental mathematics when it comes to series tests, but not relational mathematics.

Perhaps students develop an instrumental understanding rather than a relational understanding because of the structure of the course. For example, because students aren't given formal proofs of series tests and aren't given the series tests on the exams, students appear to spend much of their time studying by trying to memorize the series tests. Maybe if they had a better relational understanding of why the series tests were true and under what circumstances, students would be better equipped to apply these tests to problems on quizzes and exams.

Why might relational understanding be helpful to students? Consider the difficulty students had choosing an appropriate test. Students would start working through problems using tests that would be inconclusive, and then they would keep choosing a new test until they found one that was conclusive. Sometimes, they would use a test that would not answer the question (such as using a root test on a problem with a geometric series). Learning relational mathematics might aid students in choosing an appropriate test. For example, if students understood how the ratio test worked they might avoid trying to use it on problems that involve rational functions. In other words, if students were learning relational mathematics, students would know quickly which tests won't work with certain functions and would have more success choosing an appropriate test.

Although the examples in the preceding paragraphs focus on relational understanding to aid in choosing and using appropriate tests, relational understanding could help students with other misconceptions. For example, relational understanding would help students identify the difference between a sequence and a sequence of partial sums, which would aid students in using correct notation. Relational understanding would also help make it clear why series tests only tell us if a series converges, and not what the series converges to. Relational understanding would make it clear that a statement and its contrapositive are logically equivalent (possibly using truth tables), helping students identify the contrapositive of the nth term test.

Contributions to Existing Literature

The purpose of this section is to describe how the results of the current study contribute to the existing literature. This section is broken into four parts: general misconceptions of sequences and series, function, limits, and differentiation and integration.

General Misconceptions of Sequences and Series. The results of this study contribute to the existing literature on student misconceptions of sequences and series in several ways. First, this study expands on the results of Nardi and Iannone's (2001) study that students have trouble accepting that comparison tests can be inconclusive by showing that students also had difficulty accepting that the ratio test can be inconclusive. Student exam work showed that some students would get a value of 1 in the ratio test and conclude that the series converged to 1, rather than stating that the ratio test is inconclusive.

Second, students had trouble determining which series test to use when approaching a problem about series convergence. This trouble includes student difficulties identifying a geometric series, and instead using a root test to determine convergence.

Students also had difficulty identifying the contrapositive of the nth term test, first reported in Earls (in press). Existing literature describes the difficulties students and teachers have with the logical equivalence of a statement and its contrapositive in geometry (Gregg, 1997). The current study shows that students also have difficulty with this equivalency in second semester calculus.

Differentiating between the limit of a sequence and the sum of a series was also difficult for students. More specifically, students have difficulty understanding the difference between a sequence and a sequence of partial sums. Existing literature noted that students have difficulty understanding definitions when studying sequences (Alcock & Simpson, 2004; Alcock & Simpson, 2005; Roh, 2008), and students have a difficult time reading and understanding symbols in mathematics (Marjoram, 1974; Chirume, 2012; Earle, 1977). The results of this study contribute to these findings by showing that students specifically have trouble with limit and series notation, and that they don't think about definitions when approaching problems in second semester calculus. More specifically, students do not appear to think of a series as a limit of partial sums.

Students sometimes failed to check that the assumptions of a series test were satisfied before using this test. In the exam data, students neglected to check the assumptions of the integral test were satisfied, and in the interviews a student failed to test that the assumptions of the direct comparison test were satisfied. An ERIC search did not reveal any literature about checking assumptions in series tests, indicating that this result could be a new contribution to the literature.

Finally, students had misconceptions about what the conclusions of series tests told them. There are several examples of students thinking that a series test told them what the series converged to, rather than just whether the series converged. These examples show that in addition to the fact that students have a hard time accepting series tests can be inconclusive (Nardi and Iannone, 2001), they have a hard time knowing what to do when they get a numerical value from a series test.

Function. Sajka (2003), and Carlson and Oehrtman (n.d.) noted that students had difficulty understanding function notation. Sajka (2003) found that one student did not understand the notation f(3), associating it with the zero of the function, and thought of f(x), f(y), and f(x+y) as three different functions. Carlson and Oehrtman (n.d.) found that students evaluated f(x + a) as f(x) + a. The difficulty students had with summation and limit signs equating $\sum a_n$ and $\lim_{n\to\infty} a_n$ in this study is consistent with Sajka's (2003) and Carlson and Oerhtman's (n.d.) findings. Since students are looking at limits and summations of functions, students were having difficulty using notation that indicates an operation performed on a function. For example, several students in this study did not indicate the function that they were taking the limit of, writing $\lim_{n\to\infty} = 1$.

Students also had difficulty determining the end behavior of trigonometric sequences and rational sequences. For trigonometric sequences, this is consistent with Weber's (2005) findings on the difficulties students had determining when and why the sine function decreases. If students are unsure of where the sine function is decreasing, it might be hard for them to understand the oscillating nature of the sine function. For rational sequences, students tried to argue about the end behavior by incorrectly comparing the numerator and denominator of a rational function. This also adds to the existing literature on student difficulty with functions by showing that students argue incorrectly about the end behavior of rational functions. More specifically, students argued that a sequence diverges because the numerator grows faster than the denominator for a function such as $\frac{n+\ln n}{n}$.

Students also showed difficulty simplifying rational functions and exponential functions. Students cancelled through a sum when simplifying rational functions, and had difficulties with properties of exponential functions. These findings are consistent with the findings on functions by Makonye and Khanyile (2015), who found that students had difficulties simplifying rational functions in general and cancelled through sums and Cangelosi et al. (2013), who found that students had difficulty simplifying exponential functions that had a negative sign, the same issue that was seen in the current study.

Limits. In addition to confusing limit notation and summation notation, one student pointed out that he didn't use limit notation on his homework because he found it to be a "pain." This is consistent with William's (1991) work where students didn't feel the need for a formal definition of limit because they didn't need it to solve problems by showing that at least one student didn't see the need for limit notation because he didn't think he needed it to solve problems. Further research is needed to see if more students also don't see the need for proper limit notation.

Student work also confirmed the findings of Cottrill et al. (1996) that some students, when asked to find the limit of a function at a point a, instead gave the value of the function at a. Student exam work showed that students would "plug in" infinity to evaluate a limit at infinity, meaning that the student thought evaluating a limit at infinity is the same as "plugging in" infinity. More specifically, it appears students may think of infinity as a number, or perhaps students are unsure how to show their work in problems that deal with infinity.

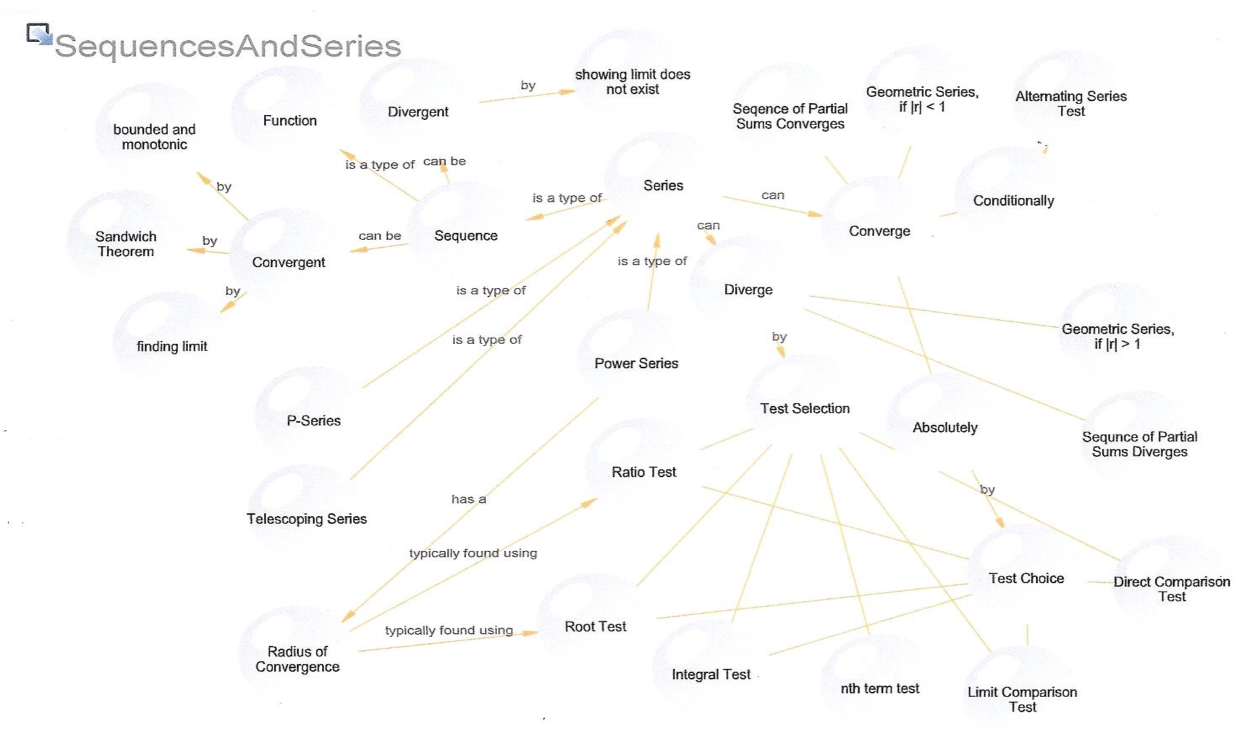
Differentiation and Integration. Difficulties with differentiation showed up in student work that involved using L'Hôpital's rule. For example, one student stated that when $f(n) = n$, that $f'(n) = \frac{1}{n}$. Orton (1983) found that students had difficulties differentiating basic functions, such as differentiating ${}^{2}/_{x^2}$. This study confirms that students also have difficulty finding derivatives of basic linear functions.

Although problems were given where the integral test could have been used, the integral test was rarely used on the exam or in interviews, and aside from failing to check that the assumptions of the integral test were satisfied, students seemed to integrate correctly, which would indicate that integration was not a problem for most students. However, it is worth noting that, on the multiple-choice assessment, students that expected an 'A' in the course performed significantly better than students that expected a 'B' or a 'C' or lower in the course as stated in the quantitative results chapter. Since students were asked about their expected grade prior to their assessments on sequences and series, their expectation of course grade comes entirely from their performance on integration. Consequently, on the multiple-choice assessment on sequences and series, it seems that students that did well on integration assessments greatly outperformed students that did not do as well on integration assessments. Further research is needed to investigate the correlation between success with sequences and series and success with integration.

Misconceptions and the Optimal Concept Map

Recall that, in the context of this study, an *optimal concept map* is a map that is representative of what the ideal second semester calculus student would know about sequences, series, and all prerequisite skills upon completion of the course. The optimal concept map can be found in Appendix A, and is also given in figure 6.3.

Figure 6.3: Optimal Concept Map



Each of the misconceptions described in Table 6.2 indicate potential gaps in the concept images of the second semester calculus students that participated. This section discusses how an understanding of the nodes in the optimal concept map can help students address these misconceptions. In the sections below, each misconception is listed, along with a description of how an understanding of nodes in the optimal concept map would help students avoid these misconceptions. The nodes for p-series and telescoping series were added to the map as a result of the current study.

Notation. Using proper notation and terminology when dealing with sequences and series was about determining when to use limit notation versus summation notation. Sequences and series each have their own node in the optimal concept map, and students should recognize that a sequence is a list of terms, while a series is a sequence of partial sums. When referring to a sequence, students should know to use the limit sign, and they should use a summation sign when referring to a series.

L'Hôpital's Rule. L'Hôpital's Rule can be applied to some functions with a certain behavior in their numerators and denominators. A student with a strong understanding of rational functions, a node in the optimal concept map, would know when and how to apply L'Hôpital's Rule. More specifically, if a student has a relational understanding of rational functions, he will be able to determine if the function is of the form $\frac{"\infty"}{"\infty"}$ or $\frac{0}{0}$.

Using and Choosing Appropriate Tests, Assumptions for Series Tests, and Conclusions of Series Tests. Knowing which test to use, why to use it, and the conclusions of that test are something a student with a strong understanding of each test would be able to do. Each test has its own node in the optimal concept map, and students need to understand which test is best to use in each circumstance. Understanding these nodes means knowing which test is best to use when given a particular series.

Trigonometric Sequences. Sequences are types of functions, and students should be familiar with trigonometric functions from both precalculus and first semester calculus. Note that function is a node in the optimal concept map. Students need to understand the end behavior of trigonometric functions. More specifically, students need to understand the oscillating end behavior of the sine and cosine functions in order to investigate the convergence of sequences that deal with the sine and cosine functions.

Algebraic Simplification of Rational and Exponential Functions. The simplification of rational functions and exponential functions are typically seen in a precalculus course, if not sooner, and reinforced in a first semester calculus course. A student with an understanding of rational and exponential functions should be able to simplify these functions. More specifically, students need to know when a rational function can be simplified, and how to simplify an exponential function with a negative sign. Again, function is a node in the optimal concept map.

Contrapositive of the Nth Term Test. While the logical equivalence of a statement and its contrapositive is a prerequisite skill that students might be expected to have entering a second semester calculus course, they specifically need to apply this reasoning to the nth term test. In particular, students should know that, if a series converges, then the limit of the sequence must be equal to 0. The nth term test is a node in the optimal concept map, and students need to know what the nth term test says and its contrapositive.

Recognizing Geometric Series. When asked to find the convergence of a series, it is important for students to be able to recognize series that are geometric because students can determine not just whether the geometric series converges, but also what it converges to. Geometric series is a node in the optimal concept map, and students need to identify ratios of transcendental numbers.

Limitations

There are several limitations to the results obtained in this study. First, as is true with many qualitative studies, it is hard to generalize the qualitative results to a larger population. Students that participated in this study all came from the same university. Consequently, any prerequisite skills students might be lacking doesn't necessarily describe all students' high school preparations, or early university preparations.

Second, in terms of quantitative results, there were too few students to determine any statistical significance among individual answer choices. There were also too few students in the sample that took precalculus at this institution to determine if they performed significantly worse than their peers.

Third, students were not asked any questions in the exam, the interviews, or the multiple-choice assessment that involved knowing a sequence can converge because it is bounded and monotonic or investigating the convergence of a telescoping series using the sequence of partial sums. Consequently, this study cannot comment on student understanding in these areas that are part of the typical calculus II curriculum.

Finally, due to the timing of the study in relation to the second semester calculus course at this institution, it was not possible to ask students questions about Taylor Series, the final topic covered in a second semester calculus course at this University. The interviews, exam data, and multiple-choice assessment all had to take place before Taylor Series were covered. Consequently, this study does not contribute to research on student understanding in this area.

Implications for Teaching and Curriculum

In addressing the third research question, an argument was given about achieving a balance between relational understanding and instrumental understanding. This section discusses the ways in which teachers and second semester calculus curriculum could emphasize relational understanding.

Textbooks and teachers could focus more on graphical depictions of sequences and series. Very few students in this study looked at graphs of the sequences or series that they were trying to determine the convergence of. A focus on graphs or visual representations might also help students identify the difference between a sequence and a sequence of partial sums.

Though students may not possess the skills necessary to formally prove theorems about series tests, informal arguments might help students see why the conclusions of series tests are true. In addition, this might help students avoid thinking that the value of a series test tells them what the series converges to. Visual representation can also be a part of the informal arguments.

Perhaps teachers can also provide better motivation for the importance of sequences and series. For example, calculators can compute the value of sin 1 by computing the Taylor polynomial of degree n (for n large) and then plugging the value 1 into the polynomial. Since many students in a second semester calculus course are in STEM fields, teachers can discuss why sequences and series are important in certain disciplines, such as engineering.

Implications for Further Research

This study has raised new questions that future research can investigate. Future studies could use the optimal concept map and the list of misconceptions in this study to further investigate student understanding of sequences and series. In what ways can the optimal concept map and the list of misconceptions in this study serve as a framework for further research on student understanding of sequences and series?

The difficulties students have with the function concept is well documented. Future research could determine in what ways does student understanding of function relate to their understanding of sequences and series?

Based on the misconceptions revealed in this study, more work can be done in the area of teaching and curriculum development. What new approaches to teaching can be used to develop a stronger relational understanding of series tests? What new curriculum materials can be developed that emphasize helping students avoid the misconceptions seen in this study?

Future research could also investigate student understanding of Taylor Series. What new misconceptions might be discovered when asking students questions about Taylor Series?

Exposure to sequences and series prior to calculus II seemed to have an impact on their success in the course, but it is unclear how this exposure was related to their success. In what ways does a student's prior exposure to sequences and series aid them when studying the topic in a second semester calculus course?

The difficulty students have determining whether to use limit notation or summation notation could be tied to the difficulties students have with symbols and their meanings. In what ways can the literature on semiotics help students better understand the difference between limit and summation notation?

Further research on student understanding of sequences and series can provide answers to these questions.

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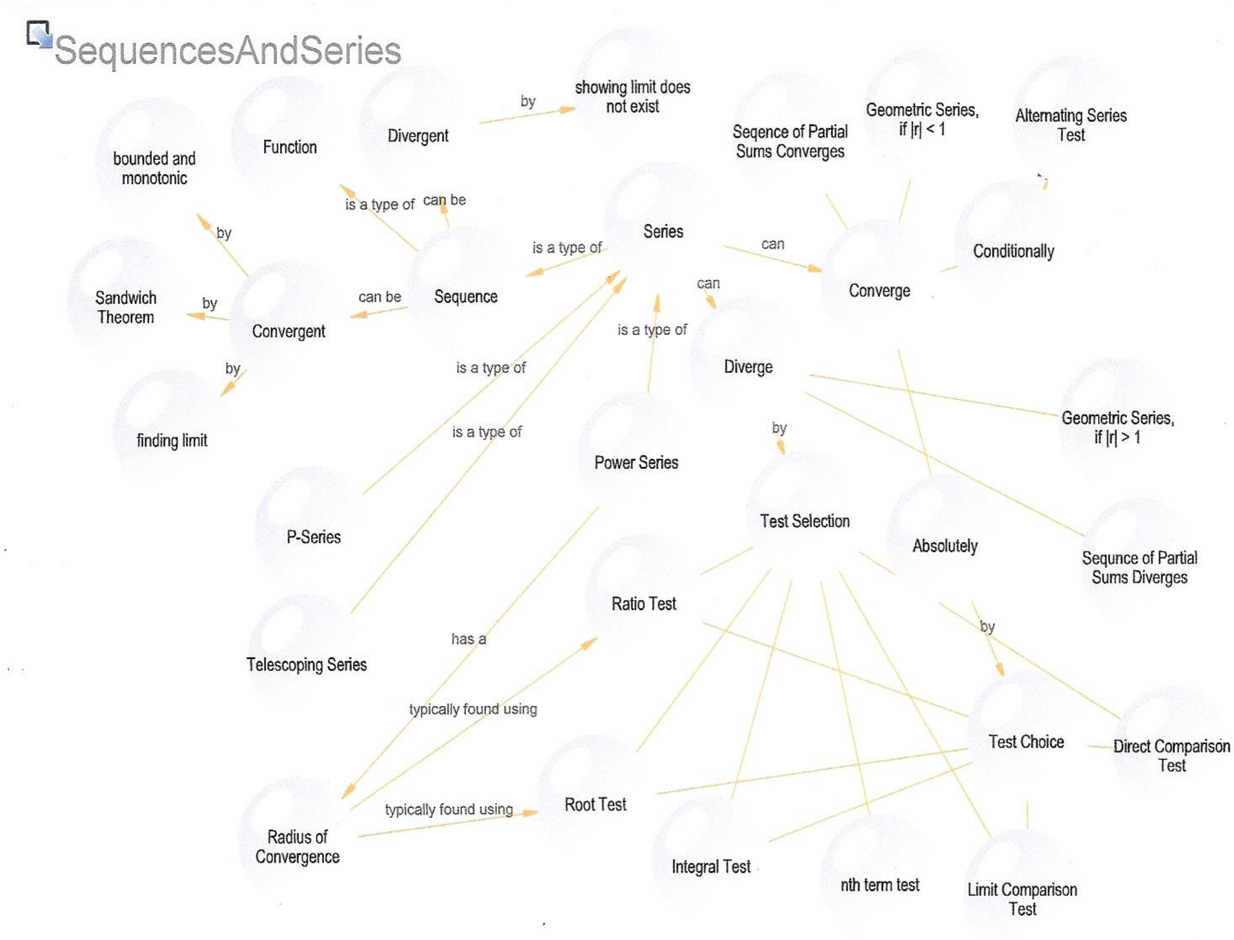
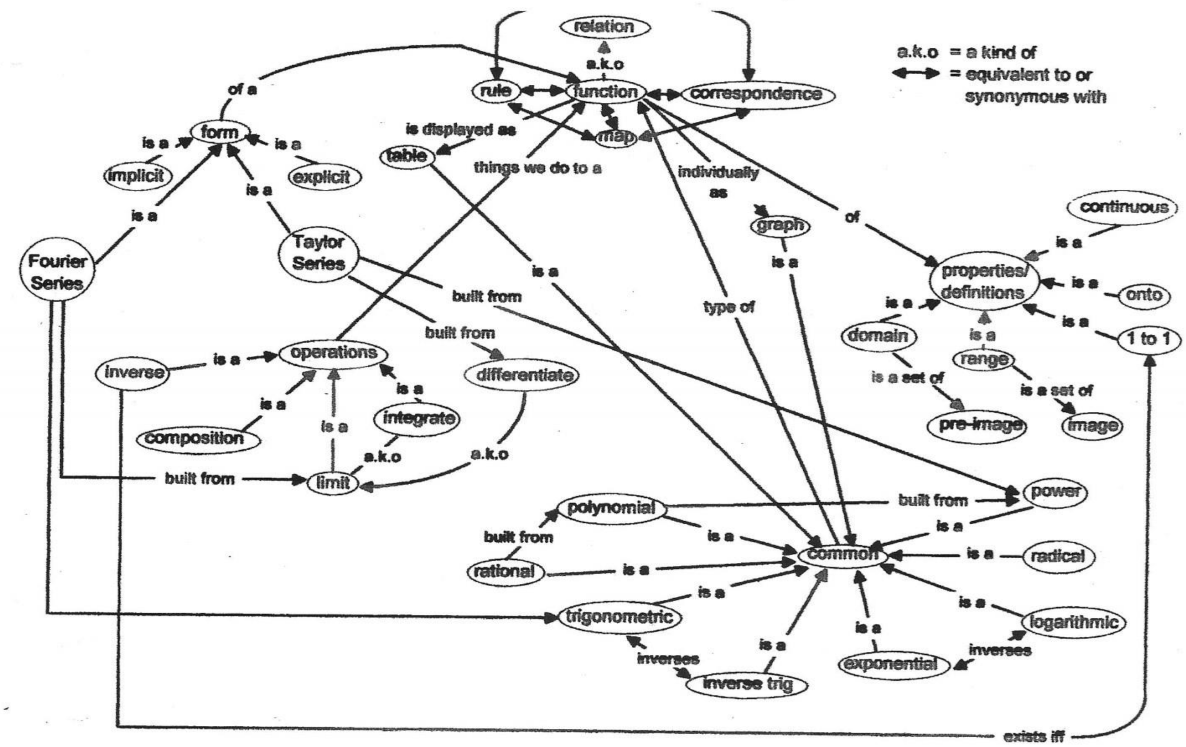
Appendix A – Optimal Concept Map



Appendix B – Expert concept map of function, adapted from Williams (1998)



Appendix C – Pilot Study Exam questions

1. (8 points each) Determine whether the following sequences converge or diverge. If the sequence diverges, specify whether it diverges to ∞ or $-\infty$ if that is the case. Find the limit of all convergent sequences. Justify your response by showing your work.

   (a)
   $$a_n = \frac{n + \ln n}{n}$$
   (b)
   $$a_n = \frac{3 + \sin n}{n}$$
   (c)
   $$a_n = \frac{\cos(n\pi)}{\pi}$$

2. (8 points each) Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work.

   (a)
   $$\sum_{n=1}^{\infty} \frac{1}{n+4}$$
   (b)
   $$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

3. (9 points each) Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work.

   (a)
   $$\sum_{n=0}^{\infty} \frac{2^{n+1}}{6^n}$$
   (b)
   $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{6^n}$$

4. (8 points each) Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work.

(a)

$$\sum_{n=0}^{\infty} e^{-3n}$$

(b)

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

5. (8 points each) Determine whether the following series converges or diverges. **Be explicit about any test you use** to justify your response. Calculate the sum of any convergent geometric series. Justify your response by showing your work.

(a)

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1}$$

(b)

$$\sum_{n=1}^{\infty} (1 - \frac{2}{n})^n$$

6. (10 points) Determine whether the following series converges absolutely, converges conditionally, or diverges. Give reasons for your answer, including any test you may have used.

(a)

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{(n+1)!}$$

Appendix D – Pilot Study Interview Tasks

Task 1

Problem: Suppose you know that $\lim_{n\to\infty} a_n = 3$. What can you say about:

$$\sum_{n=0}^{\infty} a_n$$

Suppose you know that

$$\sum_{n=0}^{\infty} b_n = 3$$

What can you say about $\lim_{n\to\infty} b_n$

Task 2

Problem: Explain your reasoning as you determine whether the series converges or diverges. If it converges, explain how you would go about finding the sum, or explain why you cannot find the sum.

$$\sum_{n=1}^{\infty} \pi^n e^{-2n}$$

Task 3

Problem: Explain your reasoning as you determine whether the series converges or diverges. If it converges, explain how you would go about finding the sum, or explain why you cannot find the sum.

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

Task 4

Problem: Explain your reasoning as you determine whether the series converges or diverges. If it converges, explain how you would go about finding the sum, or explain why you cannot find the sum.

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{(2n+1)!}$$

Task 5

Problem: Explain your reasoning as you find the values of *x* for which the following power series converges and the radius of convergence of the series.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Task 6

Problem: Suppose that $a_n = \frac{\sin\frac{(2n-1)\pi}{2}}{n}$. What can you say about $\lim_{n\to\infty} a_n$? What can you say about:

$$\sum_{n=1}^{\infty} a_n$$

Appendix E – Multiple Choice Items

1. The series

$$\sum_{n=1}^{\infty} \pi^n e^{-2n}$$

   A. Converges by the nth term test
   B. Is a divergent geometric series
   C. Is a convergent geometric series
   D. Is convergent by the root test, but is not a geometric series

2. What can you say about the series:

$$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$$

   A. Diverges by a comparison test
   B. Converges by the integral test
   C. Converges by a comparison test
   D. Both the ratio and nth term tests are inconclusive, so we cannot say whether this series converges or diverges.

3. The series

$$\sum_{n=1}^{\infty} \frac{(-4)^n}{(2n+1)!}$$

   A. Converges by the alternating series test, but is not absolutely convergent
   B. Converges absolutely
   C. Diverges by limit comparison
   D. Diverges absolutely

4. What is the interval of convergence for the following power series?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

   A. $-1 < x \leq 1$
   B. $0 < x \leq 1$
   C. Convergent for all x
   D. Divergent for all values of x

5. Suppose that $a_n = \frac{\sin\frac{(2n-1)\pi}{2}}{n}$. What can you say about $\lim_{n\to\infty} a_n$?

   A. Alternates from -1 to 1 forever
   B. $a_n \geq \frac{-1}{n}$, so by the comparison test, $\lim_{n\to\infty} a_n$ diverges.
   C. Using L'Hôpital's rule, we have $\lim_{n\to\infty} a_n = \infty$
   D. $\lim_{n\to\infty} a_n = 0$ by the Sandwich Theorem

6. Suppose that you know that

$$\sum_{n=1}^{\infty} b_n = 3$$

   What can you say about $\lim_{n\to\infty} b_n$?

   A. $\lim_{n\to\infty} b_n = \infty$
   B. $\lim_{n\to\infty} b_n = 0$
   C. $\lim_{n\to\infty} b_n = 3$
   D. We can't say anything about $\lim_{n\to\infty} b_n$

Appendix F – Multiple Choice Exam Cover Sheet

1. Please list the previous three mathematics courses you have taken prior to MATH 426, with your most recent course listed first (eg. MATH 425, MATH 418, high school precalculus)

   A__________________________________

   B__________________________________

   C__________________________________

2. Have you had any experience with sequences and series prior to entering this course, and if so, where have you seen them before?____

   ____

3. How old are you? _____________ years old

4. What year are you? (freshman, sophomore, continuing ed, etc) _____________________

5. What is your gender? ____________________________

6. Please make the most appropriate selection regarding your race:

   __Black __White __Hispanic __Asian
   __Native American __Other __Decline to Answer

7. What is your expected grade in this course?

   __A __B __C __D __F __Don't Know

Appendix G – IRB Approval

University *of* New Hampshire

Research Integrity Services, Service Building
51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

01-Apr-2015

Earls, David
Mathematics and Statistics, Kingsbury Hall
15 Gantry Street
Manchester, NH 03103

IRB #: 6206
Study: Effect of Prerequisites on Student Understanding of Sequences and Series
Approval Date: 01-Apr-2015

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved the protocol for your study as Expedited as described in Title 45, Code of Federal Regulations (CFR), Part 46, Subsection 110.

Approval is granted to conduct your study as described in your protocol for one year from the approval date above. At the end of the approval period, you will be asked to submit a report with regard to the involvement of human subjects in this study. If your study is still active, you may request an extension of IRB approval.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the attached document, *Responsibilities of Directors of Research Studies Involving Human Subjects*. (This document is also available at http://unh.edu/research/irb-application-resources.) Please read this document carefully before commencing your work involving human subjects.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie Simpson

Julie F. Simpson
Director

cc: File
Graham, Karen

Appendix H – IRB Extension Approval

University *of* New Hampshire

Research Integrity Services, Service Building
51 College Road, Durham, NH 03824-3585
Fax: 603-862-3564

03-Mar-2016

Earls, David
Mathematics and Statistics
15 Gantry Street
Manchester, NH 03103

IRB #: 6206
Study: Effect of Prerequisites on Student Understanding of Sequences and Series
Review Level: Expedited
Approval Expiration Date: 01-Apr-2017

The Institutional Review Board for the Protection of Human Subjects in Research (IRB) has reviewed and approved your request for time extension for this study. Approval for this study expires on the date indicated above. At the end of the approval period you will be asked to submit a report with regard to the involvement of human subjects. If your study is still active, you may apply for extension of IRB approval through this office.

Researchers who conduct studies involving human subjects have responsibilities as outlined in the document, *Responsibilities of Directors of Research Studies Involving Human Subjects.* This document is available at http://unh.edu/research/irb-application-resources or from me.

If you have questions or concerns about your study or this approval, please feel free to contact me at 603-862-2003 or Julie.simpson@unh.edu. Please refer to the IRB # above in all correspondence related to this study. The IRB wishes you success with your research.

For the IRB,

Julie F. Simpson
Director

cc: File
Graham, Karen